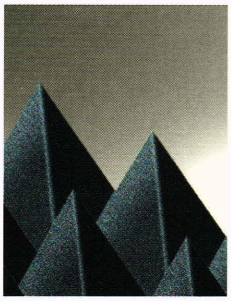


Deriving Effective Sweep Width for Intermittent Signals

Kurt T. Brintzenhofe

In search theory applications, effective sweep width is often the key parameter when determining the probability of detection against uniformly distributed targets. Typically, a sensor's effective sweep width is computed based on continuous target signals; however, almost every type of target emits at least one form of intermittently occurring signal. To determine the utility of searching for intermittent signals, a general methodology is derived that evaluates a sensor's effective sweep width using a passive, omnidirectional detection system. Under this methodology signals may be continuous, or they may be independent, intermittently occurring signals whose interarrival time distributions can be represented by any one of the family of Erlang probability distributions.

INTRODUCTION

When trying to improve a search platform's ability to detect a target by increasing its passive sensor effective sweep width, the value of using all available target signals can be significant. Even so, modeling efforts often ignore or modify certain important signals to avoid complicated mathematics. Likewise, problems in passive detection search theory tend to revolve around exploitation of continuous target signals or, in rare cases, intermittently occurring signals that are easily modeled. This article describes the use of stochastic process theory techniques to derive a searcher's effective sweep width against a target that is emitting continuous and multiple-source intermittent signals of a general nature.

The term "signal" refers here to any phenomenon produced by a target that can be detected by passive means in a given environment. As an example of an intermittently occurring signal and its effective sweep width, consider a submerged submarine's periscope occasionally breaking the water's surface while passing through a channel. Now place an observer, whose visual detection range varies with time owing to weather conditions, on a buoy in the middle of the channel. The observer cannot see either side of the channel from the vantage point of the buoy. For the observer to detect the submarine's periscope, the periscope must break the surface at least once while within detection range. If effective sweep width is calculated for this problem, the mean probability of detecting the submarine's periscope from the buoy will be the effective sweep width divided by the channel width.

To simplify calculations of the probability of detection, the actual detection laws governing a sensor's ability to detect a target in a given environment are transformed when possible to definite range (cookie-cutter) detection laws. Detection laws, as used here, include not only sensor capabilities but also environmental and signal characteristics that affect the sensor's ability to detect a target. The transformation is done so that, under the same conditions, the same number of uniformly distributed targets of identical velocity will be detected under either law.[1] The term "effective sweep width" applies to the resulting value of the transformation.

When considering passive sensor detection of a signal within a given environment, two events must occur simultaneously: the signal must occur, and it must be within sensor detection range. A sensor's effective sweep width depends on both events. The probability of a signal occurring while within detection range depends not only on how long the target remains within that range, but also on the signal's interarrival time distribution. Calculating a sensor's effective sweep width against intermittently occurring signals has until now required that the signals have exponential (Erlang type 1) interarrival time distributions. The more generalized approach developed in this article requires only that each signal of interest have an Erlang type k interarrival time distribution. Figure 1 depicts examples from the family of Erlang distributions with mean interarrival time λ^{-1}.

Independent target signals having Erlang interarrival time distributions are chosen on the basis of the Erlang distribution's relationship to the exponential distribution. Using this relationship, one can model these signals as a Markovian arrival process. Thus, the Chapman–Kolmogorov equations[2,3] can be used to derive the probability of a signal being emitted while a target passes within sensor detection range at a given relative speed. From this, a probability of detection curve that depends on the target's closest point of approach or lateral range is constructed. The area under this curve is the sensor's effective sweep width.[1,4]

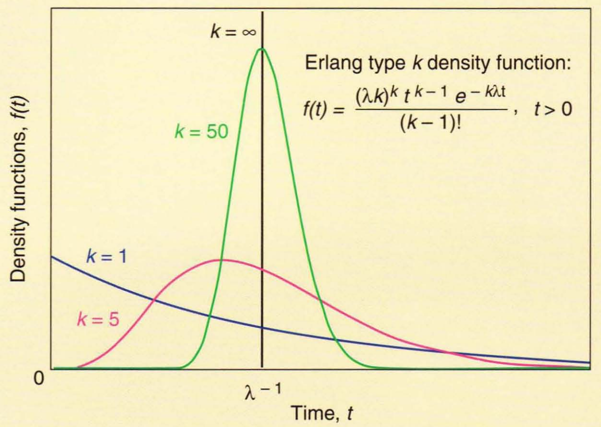


Figure 1. Sample Erlang type k distributions with mean interarrival time λ^{-1}.

BASIC CONCEPTS

Before we describe the mathematical development, three basic but important concepts must be introduced. The first, mean sensor detection range for a given environment, focuses on how temporal variations in sensor detection range resulting from fluctuations in environmental and signal strength are dealt with. Next, independence of detection events is defined. Finally, using the mean sensor detection range as a stepping stone, the discussion proceeds to the lateral range concept.

Mean Detection Range

Before we can construct effective sweep-width models, the concept of mean detection range (MDR) must be understood. In acoustics, for example, MDR is usually referred to as median detection range, probably because the random variable representing signal excess in decibels appears to be normally distributed in passive acoustics.[5] Since the mean and median of a Gaussian distribution are the same, the use of either term is correct. This section addresses the time-varying nature of a sensor's threshold detection range. Because of this variability, a sensor can demonstrate poor performance one day and outstanding performance the next. This concept occasionally gets lost in computations of the expected probability of detection.

For a continuously monitoring passive sensor, let threshold detection range x be defined as the range at which an approaching target's signal-to-noise ratio just equals the sensor's established detection threshold based on a 0.5 probability of detection and an acceptable false alarm rate. To predict sensor performance, the detection system parameters, noise level, signal level, and signal propagation characteristics must be estimated. Since the value of any factor at any particular instant can be greater than or less than its estimated value, the threshold detection range x for a given intermittent signal can be viewed as a random variable with associated density function $f(x)$ (Washburn[6]).

This article addresses only unimodal density functions representing x. More computationally time-consuming multimodal density functions can be approached similarly. For further analytic simplification, the value of the random variable x is assumed to be constant in all sensor directions at any particular moment in time. This assumed spatial symmetry facilitates representation of a "zone of detection." The zone of detection for some ith-type intermittent signal is defined as the area in which that particular intermittent signal has the opportunity to be processed as a target. The word "opportunity" is used to indicate that, even though the signal is modeled as being detected by the sensor, false alarms or other factors in the real world can cause it to be missed.

The ith-type intermittent signal is associated with its threshold detection range density function $f_i(x)$. Its MDR, which is the mean radius of the zone of detection, is simply the mean of the threshold detection range random variable x, or

$$\mathrm{MDR}_i = \int_0^{\infty} x f_i(x)dx .$$

This static representation of threshold detection range conservatively simplifies the calculations of mean probability of detection.

To illustrate concepts in this section, let $f_i(x)$ be the threshold detection range density function just defined. Define $F_i(d)$ as the cumulative distribution function of the ith-type intermittent signals, given a signal occurring at distance d from the sensor. Figure 2 provides examples of $f_i(x)$ and $F_i(d)$ using a standard normal probability density function. The MDR_i of the density function is 20 units of distance.

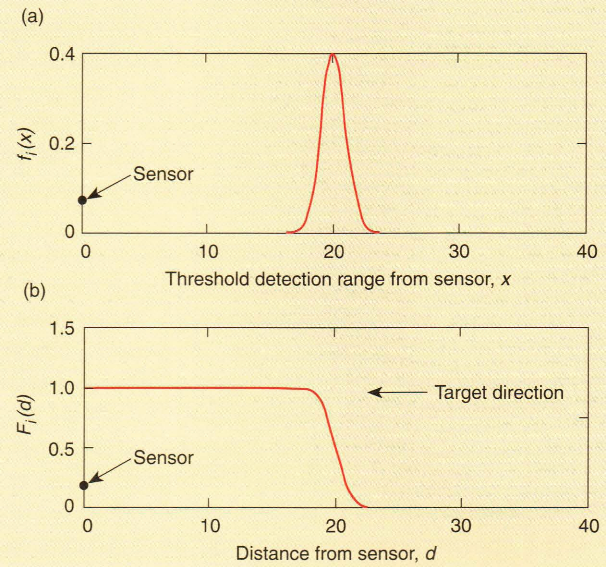


Figure 2. Examples of (a) threshold detection range density function $f_i(x)$ of the ith-type intermittent signal; and (b) cumulative distribution function $F_i(d)$ of the ith-type intermittent signal occurring at distance d from the sensor. Both examples use a standard normal probability density function with mean detection range (MDR_i) of 20 units of distance.

Independent versus Dependent Probability of Detection

Let B_1 and B_2 denote two identical intermittent signal events at the same location. If the probability P of detecting B_1 is independent of the probability of detecting B_2, then

$$P\left\{B_{1_{\text{detected}}} \middle| B_{2_{\text{detected}}}\right\} = P_{B_1|B_2} = P_{B_1} = P\left\{B_{1_{\text{detected}}}\right\} . \quad (1)$$

If Eq. 1 does not hold for the two events, then the events are dependent to some degree. In the following sections, using MDR as the range at which detection will occur, dependence between signals is modeled. If B_2 is not detected at distance d, then B_1 will not be detected at distance d. Although methodology presented later in this article would allow independence to be considered also, we avoided this approach because it may not be realistic over time frames short enough to be measured in hours.[7] Although the dependence used here is probably not entirely correct either, it is conservative, if not more realistic.[6]

Sweep Width and Lateral Range Defined

To introduce the concept of sweep width, consider a sensor and a target in motion passing the sensor. The notion of effective sensor sweep width allows a target that has *passed by* a sensor within its sweep width to be detected with probability 1 (definite range law).[1] To derive sweep width, we must define the lateral range curve concept of the target–sensor combination.

For a particular sensor and environment, a lateral range curve represents a target's cumulative probability of being detected as a function of its lateral range. As shown in Fig. 3a, lateral range is the distance between the target track line and the sensor at closest point of approach. Experimentally, we can obtain a lateral range curve if a target follows a straight course while within sensor detection range. In theory, an observer will see only whether detection occurs somewhere between points α and β for a given lateral range r, ignoring the true range at which detection occurs. After enough independent trials, the probability of detection $P(r)$ can be established for encounters having lateral range r. For intermittent events, $P(r)$ will likely increase as r decreases based on the target's increased time within the zone of possible detection.

Consider the hypothetical lateral range curve in Fig. 3b and assume that relative target–sensor motion is perpendicular to the page. If the probability density of the target crossing at point r between $-R$ and R is uniformly distributed, then effective sweep width is defined as

$$\mathrm{SW} = \int_{-R}^{R} P(r)dr . \quad (2)$$

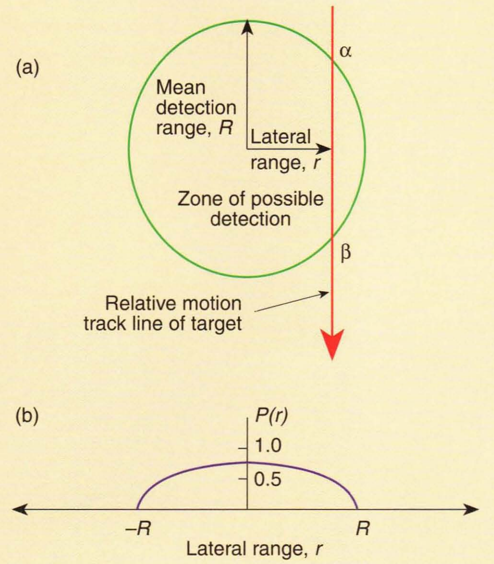


Figure 3. For a particular sensor environment, a lateral range curve represents a target's cumulative probability P of being detected as a function of its lateral range r. (a) Lateral range geometry, where α is the point at which the target enters the zone of possible detection and β is its point of departure. (b) Hypothetical lateral range probability curve.

As should be noted,

$$\mathrm{SW} \le 2R\,,$$

where R is the range after which probability of a target detection opportunity is considered insignificant.

DERIVING EFFECTIVE SWEEP WIDTH

In this section, we assume an exponential (Erlang type 1) signal interarrival time distribution when constructing a sensor's effective sweep width. The sweep width applies to a particular environment when multiple, independently occurring intermittent signals are considered. The approach taken in constructing the following models strongly resembles queuing theory models.[2] The methodology described is important, in that it allows a systematic framework within which to work.

Intermittent Signals with Exponential Interarrival Time Distributions

Under specific conditions, a sensor has an MDR_i value representing the cookie-cutter range at which a target's ith-type intermittent emissions are expected to become detectable. For expected value modeling, if an intermittent signal of interest occurs within its associated MDR, or zone of detection, it is modeled as being detected by the sensor with probability 1. Here, the probability of at least one intermittent signal occurring while the target is within the associated MDR must be determined. Using the following standard definitions,

A = the event that the target is within the sensor's zone of detection
B = the event that the target emits at least one intermittent signal
D = the event that the target emits at least one intermittent signal while within the sensor's zone of detection,

then

$$P_D = P_{AB} = P_A P_{B|A}\,.$$

In deriving P_D, the intermittent event and the detection process must be defined within a Markovian framework. This requires introducing a time-dependent detection probability factor, $N_i(t,r)$ (defined in the following paragraph). Introducing this factor causes the transition probability to be a function of time. Fortunately, the Chapman–Kolmogorov equations can still be used in solving the problem.[3]

Define $N_i(d)$ as the probability that detection of a single ith-type intermittent signal occurs when distance from target to sensor is d nmi. In other words, if d is less than the intermittent signal's MDR value, the signal can be detected. If the target begins its straight-line transit from a known position outside the sensor's zone of detection, then the target's position can be defined in terms of start time $t_0 = 0$, relative speed v, and lateral range r. If a constant relative speed v is assumed, then $N_i(d)$ can be transformed to $N_i(t,r)$. If $t_\alpha \ge t_0$ and $t_\beta \ge t_\alpha$, which represent the time that the target enters (t_α) and exits (t_β) the zone of possible detection in Fig. 3 (MDR range), then

$$\begin{aligned} N_i(d) &= \begin{bmatrix} 1, & \text{if } d \le \mathrm{MDR}_i \\ 0, & \text{otherwise} \end{bmatrix} \\ &\equiv \begin{bmatrix} 1, & \text{if } t_\alpha \le t \le t_\beta, r \le \mathrm{MDR}_i \\ 0, & \text{otherwise} \end{bmatrix} \\ &= N_i(t,r)\,. \end{aligned}$$

Let λ_i be the average arrival rate of the ith-type intermittent signals, where the arrival rate follows a Poisson distribution. Define state zero as the state where no ith-type intermittent signals have occurred within MDR_i, state 1 as the state where a single signal occurred within MDR_i, and so on. Since these states do not communicate with one another, we need only consider state zero. Letting $P_0(t)$ represent the probability of being in state zero at time t, the Chapman–Kolmogorov difference equation representing this process for M different independent intermittently occurring signals is

$$P_0(t+\Delta t,r)=\prod_{i=1}^{M}(1-\lambda_i\Delta t)P_0(t,r)+\sum_{i=1}^{M}\lambda_i\Delta t[1-N_i(t,r)]$$
$$\times\prod_{\substack{j=1,\\ j\neq i}}^{M}(1-\lambda_j\Delta t)P_0(t,r). \quad (3)$$

The right side of Eq. 3 considers two cases in which state zero can be maintained at the end of the time increment Δt. In the first case, $1-\lambda_i\Delta t$ is the probability that no ith-type intermittent signals occur during time increment Δt. The second case, represented by $\lambda\Delta t[1-N_i(t,r)]$, considers whether the intermittent event that occurred during time Δt was within sensor MDR. By setting powers of Δt in Eq. 3 that are greater than 1 equal to zero,

$$P_0(t+\Delta t,r)=P_0(t,r)-\Delta t\sum_{i=1}^{M}\lambda_i N_i(t,r)P_0(t,r).$$

After subtracting $P_0(t,r)$ from both sides, dividing by Δt, and then letting $\Delta t\rightarrow 0$, Kolmogorov's forward equation is obtained:

$$\frac{\partial P_0(t,r)}{\partial t}=-\sum_{i=1}^{M}\lambda_i N_i(t,r)P_0(t,r). \quad (4)$$

Fixing r such that $P_0(t,r)$ is dependent only on t, Eq. 4 is equivalent to

$$\frac{dP_0(t,r)}{dt}=-\sum_{i=1}^{M}\lambda_i N_i(t,r)P_0(t,r). \quad (5)$$

Equation 5 is simply a first-order homogeneous differential equation whose solution, under the given boundary conditions, is given by

$$P_0(t,r)=\exp\left[-\sum_{i=1}^{M}\lambda_i\int_0^{\infty}N_i(t,r)\,dt\right];\quad P_0(0,r)=1.$$

Since only the probability of detection as a function of lateral range r is of interest, by substituting $y=tv$ for relative target–sensor speed v, the following modification can be made:

$$P_0(r)=\exp\left[-\sum_{i=1}^{M}\frac{\lambda_i}{v}\int_{-\infty}^{\infty}N_i(y,r)\,dy\right].$$

To simplify calculation one can use the assumed symmetry of sensor detection range and the zone of possible detection's chord length $\sqrt{\mathrm{MDR}^2-r^2}$ from point α to point β of Fig. 3a, at lateral range r (where $r\leq\mathrm{MDR}$),

$$\int_0^{\infty}N_i(t,r)\,dt$$
$$\equiv\frac{1}{v}\int_{-\infty}^{\infty}N_i(y,r)\,dy$$
$$\equiv\frac{2N_i(r)}{v}\sqrt{\mathrm{MDR}_i^2-r^2},$$

$$\text{where } N_i(r)=\begin{bmatrix}1, & \text{if } r\leq\mathrm{MDR}_i\\ 0, & \text{otherwise}\end{bmatrix}. \quad (6)$$

The probability of the sensor having an opportunity to detect at least one of the M different independently occurring intermittent signals, all having exponentially distributed interarrival times at lateral range r, is

$$P_{[1,\mathrm{M}]}^{\exp}(r)=1-\exp\left[-\sum_{i=1}^{M}\frac{2\lambda_i N_i(r)}{v}\sqrt{\mathrm{MDR}_i^2-r^2}\right].$$

We can find P_D for the single sensor case by integrating over all possible uniformly distributed lateral ranges $r\in[0,R]$, such that

$$P_{D,[1,\mathrm{M}]}^{\exp}=P_{B|A}P_A=\frac{1}{R}\int_0^R P_{[1,\mathrm{M}]}^{\exp}(r)\,dr=1-\frac{1}{R}$$
$$\times\int_0^R\exp\left[-\sum_{i=1}^{M}\frac{2\lambda_i N_i(r)}{v}\sqrt{\mathrm{MDR}_i^2-r^2}\right]dr.$$

Multiplying P_D by R is one-half the effective sensor sweep width, Eq. 2. For symmetric lateral range curves, the effective sweep width for M intermittent signals can be defined as

$$\mathrm{SW}_{[1,\mathrm{M}]}=2\int_0^R P_{[1,\mathrm{M}]}^{\exp}(r)\,dr.$$

By substituting the largest MDR_i value for R, the effective sweep width for M different intermittently occurring signals having exponentially distributed interarrival times and uniformly distributed lateral ranges is

$$\mathrm{SW}_{[1,M]} = 2P_{D,[1,M]}^{\exp}\mathrm{MDR}_1 = 2\mathrm{MDR}_1 - 2 \times \int_0^{\mathrm{MDR}_1} \exp\left[-\sum_{i=1}^{M}\frac{2\lambda_i N_i(r)}{v}\sqrt{\mathrm{MDR}_i^2 - r^2}\right]dr ,$$

where $\mathrm{MDR}_1 \geq \mathrm{MDR}_2 \geq \ldots \geq \mathrm{MDR}_M$.

Multiple Intermittent Signals with Exponential Interarrival Times and a Continuous Signal

This discussion focuses on the utility of processing continuous and intermittent signals. Select a continuous signal from a target such that its associated MDR value is larger than any other continuous signal MDR from a given target. The approach used in the preceding section applies here with some modifications. First, only intermittent signal MDR values greater than the continuous signal MDR value need be considered. Second, think of the continuous signal as an intermittent signal whose average arrival rate Ψ_c is very large, such that $P_{B|A}$ approaches 1. For the maximum integer $j \in [1, M]$, such that $\mathrm{MDR}_j > \mathrm{MDR}_c \geq \mathrm{MDR}_{j+1}$, where MDR_c is the continuous-signal MDR, then for M intermittent signal types and a continuous signal, the effective sweep width is given by

$$\mathrm{SW}_{[1,M]} = 2P_{D,[1,M]}\mathrm{MDR}_1 = 2\mathrm{MDR}_1 - 2 \times \int_0^{\mathrm{MDR}_1} \exp\left[-\sum_{i=1}^{j}\frac{2\lambda_i N_i(r)}{v} \times \sqrt{\mathrm{MDR}_i^2 - r^2} - \frac{2\Psi_c N_c(r)}{v}\sqrt{\mathrm{MDR}_c^2 - r^2}\right]dr ,$$

where $\mathrm{MDR}_1 \geq \mathrm{MDR}_2 \geq \ldots \geq \mathrm{MDR}_M$.

Multiple Intermittent Signals Having Erlang Interarrival Time Distributions

The preceding section constructed the model that allows the effective sweep width to be derived for intermittent signals having exponentially distributed interarrival times. In the real world, the interarrival times of the intermittent signals may not always exhibit a Poisson arrival rate. This section constructs the effective sweep-width model for Erlang-distributed interarrival times. Since an Erlang type 1 distribution is an exponential distribution, this model is more general than those derived previously. To help readers unfamiliar with stochastic processes, the model is presented after the derivation of the more specific and relatively simple exponential distribution model.

An Erlang type k distribution is just the convolution of k exponential random variables. For example, an Erlang type 2 density function equals the convolution of two exponential density functions, or

$$f(t) = 4\lambda^2 \int_0^t e^{-2\lambda(t-y)}e^{-2\lambda y}dy = 4\lambda^2 te^{-2\lambda t} .$$

More intuitively, an Erlang type k probability density function is a probability density function representing the sum of k exponential random variables with mean interarrival time λ^{-1}/k. From the strong law of large numbers, as k tends to infinity, the variance of $f(t)$ will tend to zero. In other words, as k becomes larger, the length of each interarrival time between signals will become more predictable (deterministic).

This relationship between the Erlang and the exponential distribution is exploited to describe signal arrivals as a series of identical phases within each state.[2] As before, state zero is the state of interest. While in state zero, which implies that a signal has not occurred within sensor MDR, the interarrival time of the signals is broken into k independent and identically distributed phases. Think of the signal as climbing a set of stairs. Once the signal gets to the top of the kth step, it can be detected if a sensor is close enough to observe it. If it is not within detection range, it goes immediately to the bottom of the stairs and begins the climb again, one step at a time, toward the top of the kth step. Each phase represents a step in the signal's progress from its last occurrence to its upcoming occurrence. The length of each phase is an exponential random variable with mean λ^{-1}/k. Hence, the sum of k phases represents the length of an Erlang type k random variable.

To determine the probability that the signal is in state zero at time t, the Chapman–Kolmogorov equations are broken into k phases. Let $P_{0,j}(t)$ represent the probability that the signal is in state zero and phase $j(1 \leq j \leq k)$ at time t, so that

$$P_{0,1}(t+\Delta t) = (1-k\lambda_i\Delta t)P_{0,1}(t) + k\lambda_i\Delta t[1-N_i(t,r)]P_{0,k}(t)$$

$$P_{0,2}(t+\Delta t) = (1-k\lambda_i\Delta t)P_{0,2}(t) + k\lambda_i\Delta t P_{0,1}(t)$$

$$\vdots$$

$$P_{0,k}(t+\Delta t) = (1-k\lambda_i\Delta t)P_{0,k}(t) + k\lambda_i\Delta t P_{0,k-1}(t) . \quad (7)$$

To solve Eq. 7, the steady-state probabilities of the k phases must be determined.

Before the target arrives within the sensor's zone of detection, the probability of a signal having occurred within sensor MDR is zero. Therefore, the process shown in Eq. 7 cannot leave state zero. When $P_{0,k}$ is reached, the next phase transition must return the model to phase 1 of state zero, not to state 1. This can be represented by

$$P_{0,1}(t+\Delta t) = (1-k\lambda_i\Delta t)P_{0,1}(t) + k\lambda_i\Delta t P_{0,k}(t)$$
$$P_{0,2}(t+\Delta t) = (1-k\lambda_i\Delta t)P_{0,2}(t) + k\lambda_i\Delta t P_{0,1}(t)$$
$$\vdots$$
$$P_{0,k}(t+\Delta t) = (1-k\lambda_i\Delta t)P_{0,k}(t) + k\lambda_i\Delta t P_{0,k-1}(t)\,.$$

For steady-state conditions, this reduces to

$$\frac{dP_{0,1}(t)}{dt} = 0 = -k\lambda_i P_{0,1}(t) + k\lambda_i P_{0,k}(t)$$
$$\frac{dP_{0,2}(t)}{dt} = 0 = -k\lambda_i P_{0,2}(t) + k\lambda_i P_{0,1}(t)$$
$$\vdots$$
$$\frac{dP_{0,k}(t)}{dt} = 0 = -k\lambda_i P_{0,k}(t) + k\lambda_i P_{0,k-1}(t)\,.$$

Solving the steady-state value for each $P_{0,j}$, $j = 1,2,\ldots,k$, knowing that

$$P_0(\text{steady state}) = 1 = \sum_{j=1}^{k} P_{0,j}(\text{steady state})$$

results in

$$P_{0,j} = \frac{1}{k}\,.$$

The k steady-state values can now be used as boundary conditions at time t_α (the time the target enters the zone of possible detection in Fig. 3a) when solving Eq. 7. For a target emitting a single type of intermittent signal having an Erlang type k interarrival time distribution, the probability of a signal not occurring within the sensor's zone of detection by time t while passing at lateral range r is

$$P_0(t) = \sum_{j=1}^{k} P_{0,j}(t)\,,$$

where

$$P_{0,1}(t) = \frac{1}{k}\exp\left[-k\lambda_i \int_0^t N_i(t,r)\,dt\right]$$

$$P_{0,2}(t) = \lambda_i \int_0^t N_i(t,r)\,dt \times \exp\left[-k\lambda_i \int_0^t N_i(t,r)\,dt\right] + P_{0,1}(t)$$

$$\vdots$$

$$P_{0,j}(t) = \frac{k^{j-2}\left[\lambda_i \int_0^t N_i(t,r)\,dt\right]^{j-1}}{(j-1)!} \times \exp\left[-k\lambda_i \int_0^t N_i(t,r)\,dt\right] + P_{0,j-1}(t)\,.$$

Substituting in Eq. 6, the probability of the sensor detecting at least one of the intermittent signals as the target passes through the zone of detection at lateral range r is represented by

$$P_i^{\text{Erlang}}(r) = 1 - \exp\left[\frac{-k2\lambda_i N_i(r)}{v}\sqrt{\text{MDR}_i^2 - r^2}\right] \times \sum_{j=0}^{k-1} \frac{(k-j)\left[\frac{k2\lambda_i N_i(r)}{v}\sqrt{\text{MDR}_i^2 - r^2}\right]^j}{kj!}\,.$$

Note here that as $k \to \infty$, the interarrival time of the signal becomes deterministic with value λ^{-1}, implying that the probability of detection can be represented by

$$P_i^{\text{Erlang}}(r) = \min\left\{1, \frac{2\lambda_i N_i(r)}{v}\sqrt{\text{MDR}_i^2 - r^2}\right\}, \text{as } k \to \infty\,.$$

In general, for M different independently occurring intermittent signals, each having an interarrival time distribution that can be represented by an Erlang distribution of not necessarily the same type,

$$P_{[1,\text{M}]}^{\text{Erlang}}(r) = 1 - \prod_{i=1}^{\text{M}}\left[1 - P_i^{\text{Erlang}}(r)\right]. \quad (8)$$

Now P_D can be determined using Eq. 8 such that

$$P_{D,[1,M]} = P_{B|A,[1,M]}\, P_A = \frac{1}{R}\int_0^R P_{[1,M]}^{\text{Erlang}}(r)\, dr\,.$$

The effective sweep width can then be derived by

$$\begin{aligned}\text{SW}_{[1,M]} &= 2P_{D,[1,M]}\,\text{MDR}_1 \\ &= 2\,\text{MDR}_1 - 2\int_0^{\text{MDR}_1} \prod_{i=1}^{M}\left[1 - P_i^{\text{Erlang}}(r)\right] dr\,,\end{aligned}$$

where $\text{MDR}_1 \geq \text{MDR}_2 \geq \ldots \geq \text{MDR}_M$.

APPLICATION OF RESULTS

The preceding section provides the general form for deriving effective sweep width against intermittent signals having Erlang type k interarrival time distributions. As shown earlier, this encompasses continuously occurring signals and intermittent signals having exponential interarrival time distributions. A sensor's effective sweep width against a target, once obtained, can now be used in various closed-form search theory detection models to determine a lower bound mean probability of detection at time t.

In using the derived equations, two important presuppositions must be maintained. As implied in the introduction, the uniformly distributed target must be able to pass each sensor at all but the most extreme threshold detection ranges. If this is not possible for the particular situation being modeled, then compensation must be made when calculating effective sweep width. A further associated assumption is that no two sensors may detect a particular target at the same time.

To demonstrate use of intermittent signal sweep width equations and the assumptions mentioned, recall the example of the buoy and the submarine periscope presented in the introduction. Instead of one observer, consider two observers, each on a buoy, such that a line passing through both buoys is perpendicular to the channel boundary. The buoys are spaced so that only one observer can detect the target as it passes parallel to the channel boundaries. Also, as before, neither observer can see the channel boundaries from the buoy because of environmental conditions. Now let the target of interest be a rowboat drifting in the night down the channel at 4 kt. Detectable signals originate from a faint white light hanging from the boat's stern and from an occupant who occasionally smokes a cigarette. For the observers (who are wearing onmidirectional infrared sensors), the MDR of the faint white light is given as 1.5 nmi, the MDR for the cigarette being lighted is 3 nmi, and the MDR of the burning cigarette is 1 nmi. Since the continuous white light has a greater MDR than the burning cigarette, the signal from the burning cigarette need not be considered. Let the interarrival time distribution of the signal from the cigarette being lighted be represented by an Erlang type 10 density function having a mean interarrival time of 1 h. Let $P_1(r)$ and $P_2(r)$ represent, respectively, the probability of detecting the white light and the probability of detecting the cigarette being lighted when the track of the boat passes a sensor at lateral range r. For a channel width of 20 nmi, the mean probability of an observer detecting at least one of the two target signals is

$$\begin{aligned}\frac{2\,\text{SW}_{[1,2]}}{20\text{ nmi}} &= \frac{2}{20}\left\{6 - 2\int_0^3 \left[1 - P_1(r)\right]\left[1 - P_2(r)\right] dr\right\} \\ &= \frac{2 \times 5.36}{20} = 0.536\,,\end{aligned}$$

where

$$P_1(r) = 1 - \exp\left[\frac{-20000 N_1(r)}{4}\sqrt{1.5^2 - r^2}\right]$$

$$P_2(r) = 1 - \exp\left[\frac{-10 \times 2 \times 1 N_2(r)}{4}\sqrt{3^2 - r^2}\right] \times \left\{\sum_{j=0}^{9}(10-j)\frac{\left[\frac{20N_2(r)}{4}\sqrt{3^2 - r^2}\right]^j}{10j!}\right\}.$$

If the rowboat occupant did not smoke, the mean probability of detecting the rowboat's presence would be 0.3.

CONCLUSION

Calculating sweep width using the derived equations can be formidable if integral tables and simple numerical methods are used. The easiest approach is to use mathematical software packages such as MATHCAD or MATHEMATICA. One might reasonably ask why a Monte Carlo-type computer program simulating a target emitting continuous and intermittent signals is not used, thereby reducing the approach taken here to a mathematical exercise. This question has two answers. First, it is always good to know what to expect before using computer program-generated output. Too often the output is unintentionally misused, or simulation within the algorithm is incorrect. Therefore, having a "back of the envelope" means to calculate the expected effect of monitoring various known signals under various detection schemes is important. Second, using closed-form equations as derived here rather than repetitive, event-driven calculations within a simulation can significantly reduce the time and cost required to produce an answer.

REFERENCES

[1] Koopman, B. O., *Search and Screening*, OEG Report 56, Operations Evaluation Group, Office of the Chief of Naval Operations (1946).

[2] Gross, D., and Harris, C. M., *Fundamentals of Queuing Theory*, John Wiley & Sons, New York (1985).

[3] Cox, D. R., and Miller, H. D., *The Theory of Stochastic Processes*, John Wiley & Sons, New York (1965).

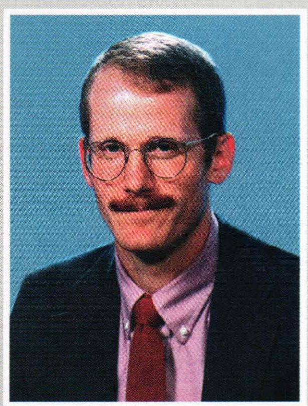
[4] Operations Analysis Study Group, U.S. Navy, Annapolis, *Naval Operations Analysis*, Naval Institute Press, Annapolis, MD (1977).

[5] Urick, R. J., *Principles of Underwater Sound*, 3rd Ed., McGraw Hill, New York (1983).

[6] Washburn, A. R., *Search and Detection*, 2nd Ed., Military Applications Section, Operations Research Society of America, Baltimore, MD (1989).

[7] Weston, D. E., "Independence in Sonar Observations," *J. Acoust. Soc. Am.* **85**(4), 1612–1616 (1989).

THE AUTHOR



KURT T. BRINTZENHOFE received a B.S. degree in mathematics from Oregon State University in 1982. From 1982 to 1988 he served as a junior officer onboard a Navy fast attack submarine and as a project coordinator/naval liaison officer within the Pentagon. In 1990, Mr. Brintzenhofe earned an M.S. degree in operations research and applied statistics from George Mason University. He joined APL in 1991. His interests include tactical utility assessment of acoustic systems, search theory applications, and strategic deterrence issues from a mathematical perspective.