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A GENERAL EXPLANATION OF THE QUASI-UNIVERSAL FORM OF THE SPECTRA OF WIND-GENERATED GRAVITY WAVES AT DIFFERENT STAGES OF THEIR DEVELOPMENT

After almost 30 years, the 1958 Phillips concept of an upper limit asymptote to the spectrum independent of wind stress is no longer tenable.[1,2] However, all observed wind-wave spectra (both frequency and wavenumber spectra), including the most recent observations, still have an extremely remarkable feature: the rear faces of all spectra, from short-fetch laboratory conditions to fully developed seas, lie practically on one line, or, more precisely, inside a very narrow envelope in a plot of spectral energy density versus wavenumber or frequency.

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Figure 1 and Table 1 (both from Ref. 6, p. 147) clearly demonstrate the existence of the envelope, in which the slopes of rear faces of individual spectra of wind-generated waves are not too different for both very "young" and very "old" waves (the differences in their amplitudes can be three orders of magnitude). Notice that the envelope in Fig. 1 lies along one line only because both spectral densities and frequencies were normalized on wind speed.

Figure 2 is the Walsh family of nondirectional individual wavenumber spectra obtained from integrating the two-dimensional spectra azimuthally. The fetch in the downwind direction was about 300 kilometers. The Walsh observations were made by a surface contour radar (see the article by Walsh et al., this issue) measuring the evolution of the directional wave spectrum with fetch. Again one can see a clearly defined envelope of rear faces (with anomalous behavior for only one of them). Notice that the envelope is again narrow even though the spectra were not normalized with respect to wind speed, since all spectra correspond to the same wind conditions.

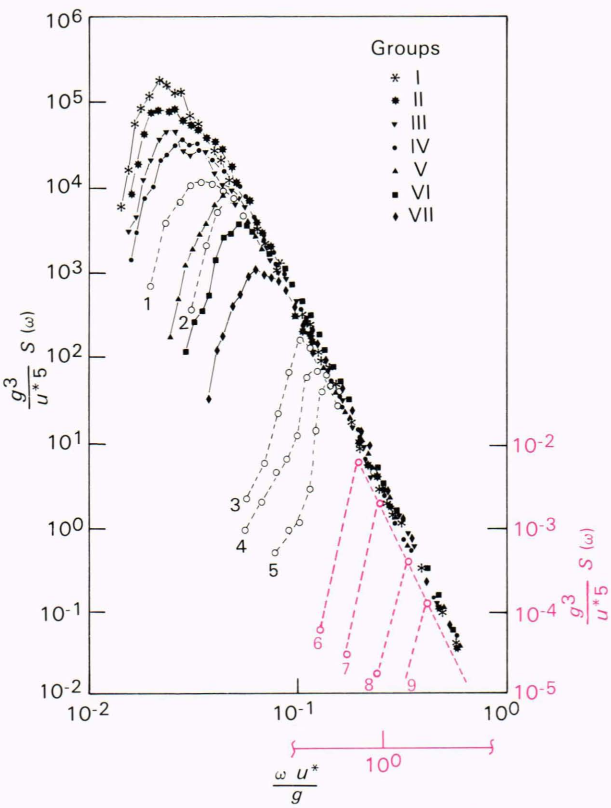

Figure 1—Dimensionless averaged spectra of wind waves for different values of nondimensional total energy, $\tilde{\sigma}_\eta^2$. Solid lines represent averaged spectra obtained from measurements in a 1965 Mediterranean expedition; numbers near spectra represented by a dotted line correspond to measurements taken by a variety of reseachers (see Ref. 6, p. 148). $S(\omega)$ is the frequency spectrum, g is gravity, and u_* is the friction velocity. u_* is in turn related to the wind velocity (U_a): $u_x \sim U_a/30$.

There are three questions in connection with Figs. 1 and 2:

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Table 1—Division of spectra into groups and characteristics of wave spectra in Fig. 1.

Group Number	*Number of Spectra in Group*	*Mean Value* $\tilde{\sigma}_\eta^2$**	*Mean Value* $\sqrt{(\Delta\tilde{\sigma}_\eta^2)^2}$	*Measurements*
I	8	5370	1040	1965 expedition of the *Akademik S. Vavilov*, Mediterranean
II	10	3470	541	
III	9	2015	392	
IV	9	811	173	
V	10	552	74	
VI	6	318	60	
VII	6	116	37	
1	1	570		Kitaigorodskii (Ref. 6, p. 148)
2	2	380		
3	5	21.2		
4*	5	12.2		
5*	3	7.3		
6	3	6.1×10^{-2}		
7	1	2.6		
8	1	6.3×10^{-3}		
9	1	2.1		

*Measurements are from Ref. 3.
**Total nondimensional wave energy, $\tilde{\sigma}_\eta^2 = g^2\bar{\eta}^2/u_*^4$ where $\bar{\eta}^2$ is the mean square surface displacement.

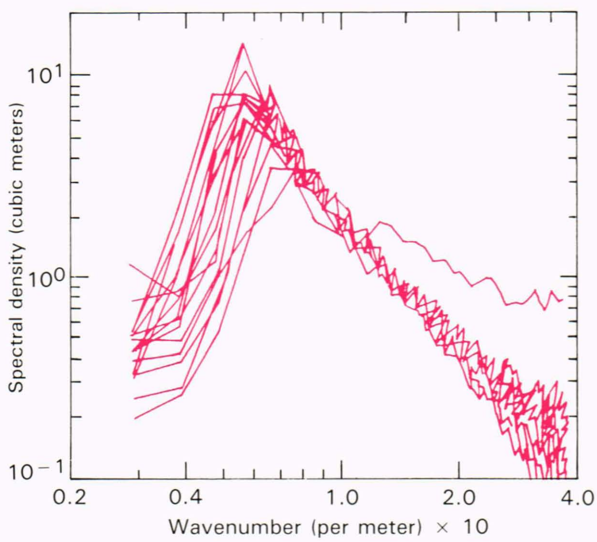

Figure 2—The family of nondirectional wavenumber spectra obtained by Walsh using a surface contour radar off the U.S. east coast on January 20, 1983 (see Walsh et al., this issue, Figs. 10 and 11).

1. Why do the slopes of the rear faces of all frequency spectra in this envelope obey roughly an ω^{-4} law,[1,2] and why does the fall-off of the high wavenumber region in nondirectional wavenumber spectra vary roughly as $k^{-5/2}$, where ω is the temporal frequency and k is the wavenumber? (Notice the difference with Phillips (1958) asymptotic spectral behavior, which exhibits an ω^{-5} and k^{-3} dependence.)
2. Why do the rear faces of the spectra have a wind-dependent form, so that the width of the envelope significantly decreases after both frequencies and spectral densities are normalized on wind speed, according to my old similarity arguments?[3]
3. Why does such a narrow envelope exist in a very wide range of frequencies (or wavenumbers), a range that covers practically all stages of wind-wave growth?

These are rather simple questions about complex phenomena. We will first formulate general answers based on only asymptotic arguments and a simplified picture of the process of wind-wave development.

In my opinion, the most difficult question among the three is the last one. Let us start with an attempt to reach an answer to the first two, probably easier questions, using the recent results of the theory of statistical equilibrium in a wind-wave field.[1,2,4,5]

In weakly nonlinear surface gravity waves, the conservation of energy and action implies that

$$\int gF(\boldsymbol{k})\, d\boldsymbol{k} = \text{const} = e$$

$$\int \frac{gF(\boldsymbol{k})}{\omega(\boldsymbol{k})}\, d\boldsymbol{k} = \int N(\boldsymbol{k})\, d\boldsymbol{k} = \text{const} = n\ ,$$

where $F(\boldsymbol{k})$ is the wave-energy spectral density, $N(\boldsymbol{k})$ is the spectral density of wave action per unit mass, g is gravity, and $\omega = (gk)^{1/2}$ is the frequency of free surface gravity waves. Outside the regions of energy input and dissipation, the spectral characteristics of the wave field are determined by the fluxes of energy, ϵ_0, and fluxes of action, ϵ_N, so that for the average (over all directions of wave component propagation) of $F(\mathbf{k})$ and $N(\mathbf{k})$ (denoted as F_K and N_K, respectively) we have the following expressions:

$$F_K = F_K(\epsilon_0, \epsilon_N, g, k)$$

$$N_K = N_K(\epsilon_0, \epsilon_N, g, k) \ .$$

There are two general properties of the process of wave–wave interactions that permit constructive inferences from the above relationships. One is related to the directions of action and energy fluxes through the wave spectrum, which are opposite.[5] If the regions of generation (or energy input from the wind) at $\Delta k \sim k_+$ and the regions of dissipation, $\Delta k \sim k_-$, are far separated, then on one end of the spectrum (the high wavenumber end),

$$F_K = F_K(\epsilon_0, g, k) \qquad \text{for } k_- > k > k_+ \ ,$$

and on the other (low wavenumber) end,

$$F_K = F_K(\epsilon_N, g, k) \qquad \text{for } k < k_+ \ .$$

The second general property of the process of weakly nonlinear resonant wave–wave interactions is that they are cubic in wave amplitude so that the above expressions can be written

$$F_K = F_K(\epsilon_o^{1/3}, g, k) \qquad \text{for } k_- > k > k_+$$

$$F_K = F_K(\epsilon_N^{1/3}, g, k) \qquad \text{for } k < k_+ \ .$$

The latter leads to the following final expressions for the wavenumber spectrum F_K and the corresponding frequency spectrum, $S(\omega)$:[1]

$$F_K = A\epsilon_0^{1/3} g^{-1/2} k^{-7/2} \quad \text{for } k_- > k > k_+ \ ,$$

$$S(\omega) = 2A\epsilon_0^{1/3} g\omega^{-4} \quad \text{for } \omega(k_-) > \omega > \omega_+(k_+),$$

and

$$F_K = B\epsilon_N^{1/3} g^{-1/3} k^{-10/3} \quad \text{for } k < k_+,$$

$$S(\omega) = 2B\epsilon_N^{1/3} g\omega^{-11/3} \quad \text{for } \omega < \omega_+(k_+) \ ,$$

where A and B are absolute constants, presumably of order one.

Now, according to my old similarity arguments,[6] the variability of statistical characteristics of the wave field with wind speed can be eliminated by a normalization of spectral densities and wavenumbers (or frequencies), as in Fig. 1. For energy and action fluxes we can write, for example,

$$\tilde{\epsilon}_0 = \frac{\epsilon_0}{U_a^{\ 3}} = f\left(\frac{gX}{U_a^{\ 2}}\right)$$

and

$$\tilde{\epsilon}_N = \frac{\epsilon_N g}{U_a^{\ 4}} = f_1\left(\frac{gX}{U_a^{\ 2}}\right) .$$

Therefore we can consider only the development of waves with fetch (or duration) for a given wind speed, which simultaneously corresponds to the family of nondimensional wave spectral characteristics for different nondimensional fetches $\tilde{X} = gX/U_a^2$, where X = fetch and U_a = wind speed.

Because the wave spectrum in any case is rather narrow, we can consider the low-frequency region ($\omega \ll \omega_+$) as corresponding to the conditions of large fetches X (asymptotically, as $X \rightarrow \infty$), the high-frequency region ($\omega \gg \omega_+$) as corresponding to short fetches ($X \ll X_+$), and the region of energy input, $\Delta\omega \sim \omega_+$, to be representative of intermediate fetches, $X \sim X_+$.

That means that for all stages of wave development the asymptotic behavior of equilibrium wave spectra for small and large fetches can be described using ω^{-4} ($k^{-7/2}$) and $\omega^{-11/3}(k^{-10/3})$ laws. This result can be a good answer to our first question of why rear faces of all observed frequency spectra have roughly an ω^{-4} slope, since the difference in powers for small (12/3) and large (11/3) fetches is very small, even though the dynamical consequences of wave–wave interactions in these conditions can be totally different.

The expressions for fluxes of energy and action offer a key to the answer for the second question: why do the rear faces of the spectra have a wind-dependent form? These expressions can be written in simplified forms[1] as:

$$\tilde{\epsilon}_0 = \frac{\epsilon_0}{U_a^{\ 3}} \simeq \text{const} = m$$

$$\tilde{\epsilon}_N = \frac{\epsilon_N g}{U_a^{\ 4}} \simeq \text{const} = m_1 \ ,$$

which leads to the following expressions for frequency spectra at large and short fetches:

$$S(\omega) = 2Bm_1^{1/3}\, U_a^{4/3}\, g^{2/3}\omega^{-11/3} \quad \text{for } X \rightarrow \infty$$

$$S(\omega) = 2Am^{1/3}\, U_a\, g\, \omega^{-4} \quad \text{for } X \ll X_+ \ .$$

This answers the second question.

Now we will attempt to answer the third and most difficult question, that is, why do the rear faces of all spectra, when properly normalized on wind speed and presented as a function of nondimensional frequency, lie approximately on a single line? Calculations have been made by Kitaigorodskii and Hansen[7] that relate to the question, and will be discussed below.

For sufficiently developed wind waves, the region of energy input can be in the middle of the rear face of the spectra, so that on both sides of ω_+ there can exist ω^{-4} and $\omega^{-11/3}$ equilibrium spectra. I have just one spectrum that gives an indication of this (Fig. 3, from Donelan[8]), and which shows that there can exist a matching frequency, ω_+, at which the two spectra are equal, so that

$$\tilde{\omega}_+ = \left(\frac{A}{B}\right)^3 \frac{m}{m_1} \quad .$$

The reliable estimate for $\tilde{\omega}_+$ is $\tilde{\omega}_+ \simeq 3$, which leads to the equality

$$A^3 m \simeq 3B^3 m_1 \ ,$$

or approximately (if A and B are of order 1)

$$m \sim 3m_1 \ .$$

The calculations both for large[9] and small[7] fetches demonstrate that these equalities are consistent with the empirical data and estimates of overall balances of energy and action. Therefore (remember in our construction $\tilde{\omega}_+$ corresponds to $\tilde{X}_+$), with certain justification we can accept the matching assumption that states that the wave spectra for constant action and energy fluxes are matching each other at some intermediate fetches $\Delta X \sim X_+$, and the constant action flux form dominates the spectrum for $X \gg X_+$, while the constant energy flux form dominates the spectrum for shorter fetches, $X \ll X_+$.

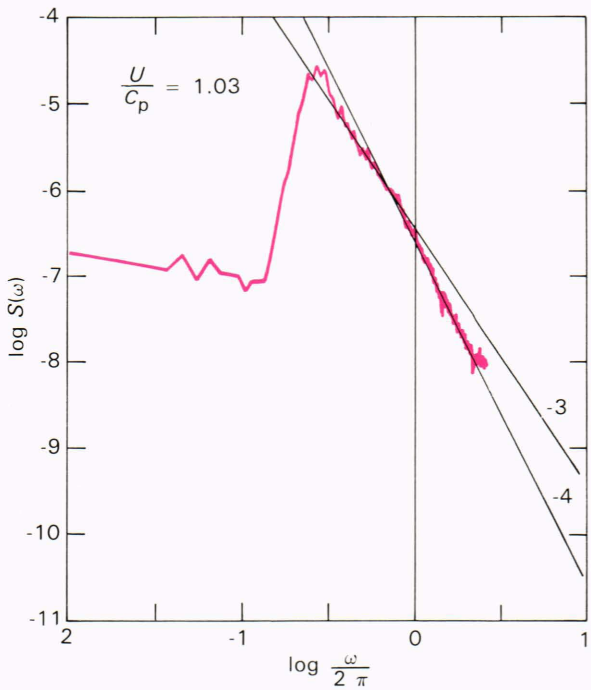

Figure 3—Frequency spectra[8] corresponding to the conditions close to the fully developed state ($U_a/c_p \approx 1$); c_p is the phase velocity at the peak. The lines show the noticeable difference in slopes on the rear face. The change in slope occurs approximately at $\tilde{\omega} = \omega U_a/g \approx 3$.

This is an answer to the third question of why the narrow envelope of wave spectra (normalized with respect to wind) exists in a wide range of nondimensional frequencies, $\tilde{\omega} = \omega U_a/g$, corresponding to practically all stages of wind-wave growth, even though the energy balance at the rear faces of wind-wave spectra can be completely different for large and short fetches.

The most critical point in our construction is the assumption about the separation of the regions of energy input and dissipation (presumably due to wave breaking). However, the asymptotic nature of our arguments must not be forgotten. More than that, Phillips[2] recently demonstrated that the final expressions for energy spectra, with energy flux toward high wavenumbers, are not very sensitive to the assumptions of how energy input from the wind, and dissipation due to wave breaking, are distributed in wavenumber space. Numerical calculations[10] confirmed the fact that for fully developed wind waves in the energy-containing region on the rear face of the wave spectrum, the energy flux reverses its direction at about twice the peak frequency.

My answers to all three questions are based on a synthesis of recent and basic achievements in our understanding of wind-wave dynamics.

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