

# THE PHYSICAL BASIS FOR ESTIMATING WAVE-ENERGY SPECTRA WITH THE RADAR OCEAN-WAVE SPECTROMETER

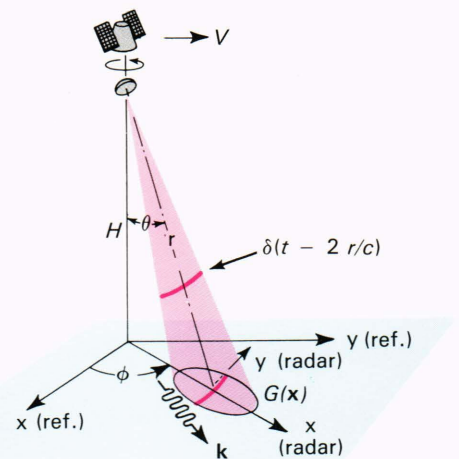
A solution for the spatial spectrum of sea-surface reflectivity for near-nadir-viewing microwave radars previously given by the author is rederived using geometrical optics. The new derivation makes somewhat clearer the physical basis for the radar ocean-wave spectrometer measurements of ocean-wave directional spectra.

## INTRODUCTION

Jackson<sup>1</sup> obtained a solution for the reflectivity modulation spectrum of the sea surface for near-nadir-viewing microwave radars based on physical optics in the high-frequency limit. Physical optics was used mainly to account for the Rayleigh fading (or speckle noise component) in the backscattered signal resulting from waveform coherency. The actual reflectivity modulation spectrum, absent the fading noise component, could as well have been derived from geometrical optics directly. Because of the potential importance of this scattering solution for future spaceborne application (see the accompanying paper, "The Radar Ocean-Wave Spectrometer," in this issue), it may be worthwhile to revisit the problem and to recast it in terms of geometrical optics *ab initio*. We do this using a delta function formalism similar to that employed by Lynch and Wagner.<sup>2</sup>

## GEOMETRIC OPTICS SOLUTION

The geometry of the measurement is shown in Fig. 1. Let  $\theta$  be the incidence angle,  $\phi$  the azimuth angle, and  $r$  the range from the radar to points  $\mathbf{r} = (\mathbf{x}, z) = (x, y, z)$  in the vicinity of the mean sea surface  $z = 0$ . Let the antenna gain pattern projected onto  $z = 0$  be denoted  $G(\mathbf{x})$ . We assume the sea surface  $z = \zeta(\mathbf{x})$  to be a homogeneous, random process in  $\mathbf{x}$  with correlation length scales (dominant wavelengths and crest lengths) small compared to the antenna footprint dimensions. (Note that the explicit time dependence of  $\zeta$  is of no concern in the present application, and so it is ignored.) Let the power envelope of a short pulse transmitted at  $t = 0$  be approximated by the Dirac impulse function  $\delta(t)$ . The power  $W(t)$  backscattered to the receiver from the sea



**Figure 1**—ROWS measurement geometry. Short transmitted pulses are approximated by the impulse function  $\delta$ . The broad electromagnetic wavefront on the surface acts to resolve surface waves with wave vectors  $\mathbf{k}$  lying in the plane of incidence.

surface is given by a form of the radar range equation appropriate to extended, noncoherent targets:<sup>3</sup>

$$W(t) = \frac{\lambda^2}{(4\pi)^3} \int \frac{\delta(t - 2r/c) G^2(\mathbf{x}) \eta(\mathbf{x}, z) d\mathbf{x} dz}{r^4} \quad (1)$$

Here,  $\lambda$  is the electromagnetic wavelength,  $c$  is the speed of light, and  $\eta(\mathbf{x}, z)$  is the reflectivity density per unit volume  $d\mathbf{x} dz$  of the region surrounding the vertically distributed sea surface. (In Eq. 1 as well as in the following, the look angle dependence of  $W$ ,  $\eta$ , and other quantities will be understood.) According to geometrical optics, if shadowing and multiple scattering are negligible (as is the case in near-nadir microwave backscatter,  $\theta \leq 15$  degrees), then  $\eta$  can be expressed as

$$\eta(\mathbf{x}, z) = \rho \pi \sec^4 \theta \delta[\nabla \zeta(\mathbf{x}) - \mathbf{s}] \delta[\zeta(\mathbf{x}) - z] \quad (2)$$



Frederick C. Jackson is a research scientist in the Laboratory for Oceans, NASA/Goddard Space Flight Center, Greenbelt, MD 20771.

where  $\rho$  is the Fresnel reflectivity for (locally) normal incidence,  $\nabla\zeta = (\zeta_x, \zeta_y)$  is the surface wave slope, and  $s$  is the specular slope,

$$\mathbf{s} = (\nabla\zeta)_{sp} = \tan\theta (\cos\phi, \sin\phi) .$$

The correctness of Eq. 2 is adduced by the more familiar relationships that follow from it. From the sifting property of the delta function and the definition of ensemble averaging, it follows that the mean reflectivity density is given by

$$\begin{aligned} \bar{\eta}(z) &= \langle \eta(\mathbf{x}, z) \rangle = \int \eta(\mathbf{x}, z) p_{\nabla\zeta, \zeta}(\nabla\zeta, \zeta) d\nabla\zeta d\zeta \\ &= \rho\pi \sec^4\theta p_{\nabla\zeta, \zeta}(\mathbf{s}, z) , \end{aligned} \quad (3)$$

where  $\langle \rangle$  denotes ensemble average and  $p(\ )$  is the joint probability density function of surface height and slope (cf. Jackson<sup>4</sup>). The conventional surface cross section per unit horizontal area  $\sigma^\circ$  is given by<sup>5</sup>

$$\sigma^\circ = \int \bar{\eta}(z) dz = \rho\pi \sec^4\theta p_{\nabla\zeta}(\mathbf{s}) . \quad (4)$$

The cross section of an isolated specular point  $\sigma_{sp}$  is given by

$$\begin{aligned} \sigma_{sp} &= \int_{sp \in \Delta x \Delta z} \eta(\mathbf{x}, z) dx dz \\ &= \rho\pi \sec^4\theta |d\nabla\zeta/d\mathbf{x}|_{sp}^{-1} \\ &= \rho\pi |K|_{sp}^{-1} , \end{aligned} \quad (5)$$

where  $K = [\zeta_{xx}\zeta_{yy} - \zeta_{xy}^2]/[1 + (\nabla\zeta)^2]^2$  is the Gaussian curvature.

If the range,  $r$ , to the surface is large compared to the antenna footprint dimensions, it can be removed from the integration. Further, for angles of incidence removed from nadir, wavefront sphericity can be ignored and the incident wave treated as a plane wave (see Jackson<sup>1</sup> for a treatment of sphericity effects). For notational convenience, let  $x$  lie in the plane of incidence and let

$$\begin{aligned} r_t &= ct/2, \\ x_t &= r_t/\sin\theta, \end{aligned}$$

and

$$r = r_t + x \sin\theta - z \cos\theta .$$

Then the backscattered power as a function of surface range,  $x$ , becomes

$$\begin{aligned} W(x_t) &= C \int \delta(x_t - x + z \cot\theta) G^2(\mathbf{x}) \\ &\quad \times \eta(\mathbf{x}, z) d\mathbf{x} dz , \end{aligned} \quad (6)$$

where  $C = \lambda^2/(4\pi)^3 r^4$ . Since  $\eta$  is concentrated about  $z = 0$ , it follows that the average backscattered power envelope is

$$\bar{W}(x_t) = C \sigma^\circ \int G^2(x_t, y) dy . \quad (7)$$

One may consider the spectrum of backscattered power as seen through the gain pattern; or one can normalize the received power versus range by the average power envelope prior to spectrum analysis, in which case one is free to rewindow the data as desired (assuming one has digitally sampled the received power). The reflectivity modulation  $m(x, \phi)$  as a function of surface range for a given antenna azimuth is defined by

$$m(x_t, \phi) = [W(x_t, \phi) - \bar{W}(x_t, \phi)]/\bar{W}(x_t, \phi) , \quad (8)$$

where the normalization is carried out between some prescribed power points in the average power envelope, say, covering an interval  $L_x$ . If  $L_x$  is large compared to the water wavelength (or correlation scale of the waves), the precise value of  $L_x$  is immaterial. The range reflectivity modulation spectrum may be defined as

$$\begin{aligned} P_m(k, \phi) &= (2\pi L_x)^{-1} \left\langle \left| \int_{L_x} m(x_t, \phi) \right. \right. \\ &\quad \left. \left. \times \exp(-ikx_t) dx_t \right|^2 \right\rangle , \end{aligned} \quad (9)$$

where  $k = 2(2\pi f/c) \sin\theta$  is the surface wavenumber and  $f$  is the detected (video) modulation frequency. Assuming a separable gain pattern,  $G(\mathbf{x}) = G_x(x) G_y(y)$ , and assuming interchangeability of ensemble averaging and integration operations, one has

$$\begin{aligned} P_m(k, \phi) &= \frac{2\pi}{[G_y^2(y) dy]^2} \cdot \frac{1}{(2\pi)^2} \int \int d\Delta\mathbf{x} d\Delta z \\ &\quad \times A_y(\Delta y) \exp[-ik(\Delta x - \Delta z \cot\theta)] \\ &\quad \times \int dz \Gamma_{\eta\eta}(\Delta\mathbf{x}; z, z + \Delta z) , \end{aligned} \quad (10)$$

where  $A_y(\Delta y) = \int G_y^2(y) G_y^2(y + \Delta y) dy$  is the convolved lateral gain pattern,  $\Delta\mathbf{x} = \mathbf{x}^2 - \mathbf{x}^1 = (\Delta x, \Delta y)$  is the separation or lag vector in the horizontal between the two points  $\mathbf{x}^1$  and  $\mathbf{x}^2$  with the associated vertical separation  $\Delta z = z^2 - z^1$ , and where the autocovariance function of reflectivity density,

$$\begin{aligned} (\sigma^\circ)^2 \Gamma_{\eta\eta}(\Delta\mathbf{x}; z^1, z^2) &= \langle \eta(\mathbf{x}, z^1) \eta(\mathbf{x} + \Delta\mathbf{x}, z^2) \rangle \\ &\quad - \bar{\eta}(z^1) \bar{\eta}(z^2) . \end{aligned}$$

From Eqs. 2 and 3 and from the definition of an ensemble average, one has

$$[p_{\nabla\zeta}(\mathbf{s})]^2 \Gamma_{\eta\eta} = p_{\nabla\zeta^1, \zeta^1, \nabla\zeta^2, \zeta^2}(\mathbf{s}, z^1, \mathbf{s}, z^2; \Delta\mathbf{x}) - p_{\nabla\zeta, \zeta}(\mathbf{s}, z^1) p_{\nabla\zeta, \zeta}(\mathbf{s}, z^2), \quad (11)$$

where  $p(\ )$  is the joint probability density function of heights and specular slopes at the two horizontal points  $\mathbf{x}^1$  and  $\mathbf{x}^2$  separated by the lag vector  $\Delta\mathbf{x}$ . Assuming, for example, a Gaussian gain pattern,  $G_y = \exp[-\frac{1}{2}(y/L_y)^2]$ , that is broad compared to the lateral correlation scale of the waves,  $kL_y \gg 1$ , then we may take  $A_y \approx A_y(0)$ ; letting the surface wave vector

$$\mathbf{k} = k(\cos \phi, \sin \phi)$$

then Eq. 10 may be written, for arbitrary orientation of the reference axes, as

$$P_m(\mathbf{k}) = (\sqrt{2\pi}/L_y) \cdot (2\pi)^{-2} \int \exp(-i\mathbf{k} \cdot \Delta\mathbf{x}) \times \Xi(k; \Delta\mathbf{x}) d\Delta\mathbf{x},$$

where

$$\Xi(k; \Delta\mathbf{x}) = \int \int \exp(ik \cot \theta \Delta z) \times \Gamma_{\eta\eta}(\Delta\mathbf{x}; z, z + \Delta z) dz d\Delta z. \quad (12)$$

### DISCUSSION

The solution obtained here agrees with that given earlier by Jackson.<sup>1</sup> The present result was derived from a time domain representation of the backscattered power (link equation) using geometrical optics, while the original solution was derived in the frequency domain as a fourth moment of the surface scattering transfer function using physical optics and proceeding to the high-frequency geometrical optics limit. While the present result is not new, its derivation is at least perspicuous, and the physical meaning of the result should be clear (Fig. 2). The result can be viewed as the three-dimensional Fourier transform of the nonstationary, height-dependent reflectivity autocovariance function averaged over the vertical. Thus, if we let  $\mathbf{K} = (\mathbf{k}, k^3) = (\mathbf{k}, -k \cot \theta)$  and  $\mathbf{R} = (\mathbf{x}, z) = (\mathbf{x}, R^3)$ , then  $P_m$  can be expressed as

$$P_m(\mathbf{K}) = (2\pi)^{-3/2} L_y^{-1} \int d\Delta\mathbf{R} \exp(-i\mathbf{K} \cdot \Delta\mathbf{R}) \times \int dR^3 \Gamma_{\eta\eta}(\Delta\mathbf{R}; R^3).$$

We note that in a strict mathematical sense, the solution does not exist since the joint probability density function has a first-order singularity at zero lag. This

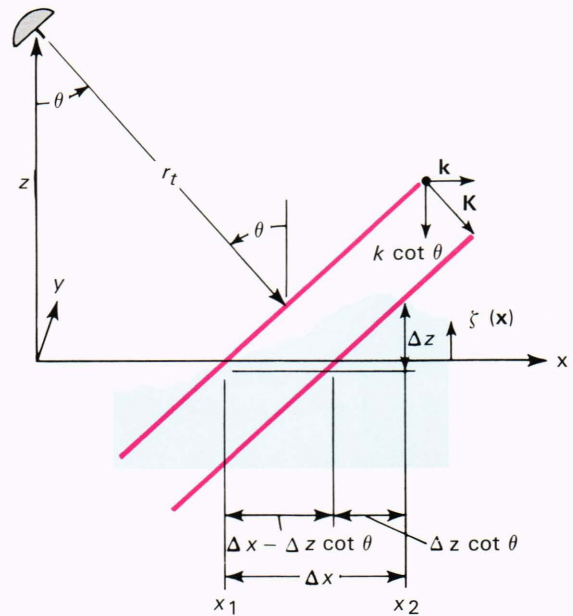


Figure 2—Radar impulses at two successive times striking the sea surface, illustrating the confounding of the horizontal coordinate  $x$  with the vertical coordinate  $z$  that occurs when the nadir angle  $\theta$  is small or when the large waves are very steep.

singularity is due to the point source approximation that is implicit in the delta function formulation. In the case of a point source, the cross section of a specular point is proportional to the reciprocal of Gaussian curvature (Eq. 5); as shown by Longuet-Higgins,<sup>6</sup> the reciprocal Gaussian curvature has unbounded variance, and so the variance of specularly reflected power is also unbounded. To eliminate the singularity, one must introduce a finite window function for the surface slopes.<sup>7</sup>

The scattering solution (Eq. 12) can also be expressed in terms of the characteristic function of the joint probability density function. This alternative solution form is better suited to an expansion in terms of the surface height and slope statistics (covariances and higher order moments) in both the Gaussian sea case and the nonlinear (weakly non-Gaussian) sea case according to Longuet-Higgins' theory.<sup>8</sup> Under the appropriate conditions, namely, that (1) the sea is approximately Gaussian, (2) the dominant wave slope  $k_0(\bar{\zeta}^2)^{1/2}$  (where  $k_0$  is the dominant wavenumber) is small compared to the root-mean-square wave slope  $[(\nabla\zeta)^2]^{1/2}$ , and (3) the incidence angle is neither too large nor too small, i.e.,  $\theta \approx$  root-mean-square slope  $\approx 10$  degrees (typically), the solution can be expanded asymptotically to yield<sup>1</sup>

$$P_m(\mathbf{k}) = (\sqrt{2\pi}/L_y) \exp(-k^2 \cot^2 \theta \bar{\zeta}^2) \times [(\cot \theta - d \ln p / ds)^2 k^2 F(\mathbf{k}) + \dots], \quad (13)$$

where  $d/ds$  is the directional derivative (in the direction of  $\mathbf{s}$ ),  $p = p_{\nabla\zeta}(\mathbf{s})$  is the slope probability density func-

tion, and  $F(\mathbf{k})$  is the directional height spectrum. The leading term in Eq. 13 (which can be shown to correspond to a simpler linear tilt model of the reflectivity modulation) has been used successfully to infer directional height spectra from high-altitude, Ku-band aircraft data (see the companion paper, "The Radar Ocean-Wave Spectrometer," in this issue; note that, to be consistent with the neglect of the higher order terms in Eq. 13, the exponential term in Eq. 13 is, as a rule, also neglected). While the linear approximation provides a fairly accurate working model, future work aimed at more precise and accurate measurements of surface-wave directional spectra (e.g., from satellite platforms such as Spectrasat) may have to consider the full scattering solution and the inverse problem it poses.

## REFERENCES

- <sup>1</sup>F. C. Jackson, "An Analysis of Short Pulse and Dual Frequency Radar Techniques for Measuring Ocean Wave Spectra from Satellites," *Radio Sci.* **16**, 1385-1400 (1981).
- <sup>2</sup>R. J. Lynch and R. J. Wagner, "Rough Surface Scattering: Shadowing, Multiple Scatter, and Energy Conservation," *J. Math. Phys.* **11**, 3032-3042 (1970).
- <sup>3</sup>R. K. Moore, "Microwave Remote Sensors," in *Manual of Remote Sensing, Vol. I*, R. G. Reeves, ed., American Society of Photogrammetry, Falls Church, Va., pp. 399-538 (1975).
- <sup>4</sup>F. C. Jackson, "The Reflection of Impulses from a Nonlinear Random Sea," *J. Geophys. Res.* **84**, 4939-4943 (1979).
- <sup>5</sup>D. E. Barrick, "Rough Surface Scattering Based on the Specular Point Theory," *IEEE Trans. Ant. Propag.* **AP-16**, 449-454 (1968).
- <sup>6</sup>M. S. Longuet-Higgins, "The Distribution of the Sizes of Images Reflected in a Random Surface," *Proc. Cambridge Phil. Soc.* **55**, 91-100 (1959).
- <sup>7</sup>F. W. Riedel, "Analytical Investigation of a Simplified Model to Study Sun Glitter Temporal Fluctuations," American Geophysical Union Spring Meeting, Washington, D.C. (May 1979).
- <sup>8</sup>M. S. Longuet-Higgins, "The Effect of Nonlinearities on Statistical Distributions in the Theory of Sea Waves," *J. Fluid Mech.* **17**, 459-480 (1963).