ON IMPROVING THE CHANCE OF HITTING THE TARGET

For many thousands of years men have been hurling stones and other ballistic missiles at targets and have used various methods of adjusting the aim so as to improve the chance that subsequent missiles will hit the target. An analytical study by A. Radcliffe and T. H. Haynes, both of APL,* suggests a method of adjusting the aim after observation of each shot. The scheme depends upon various ideas that are in general use in processing data for the estimation of means and accuracies. The basic ideas are contained in work used in 1801, but published in 1809, by Karl Friedrich Gauss. The publication, *Theoria Motus Corporum Coelestium,* was translated by the American, Charles Henry Davis, and, under the authority of the Secretary of the Navy, was published in 1857.[1]

Anyone who shoots at a target may miss it for all kinds of reasons. The two reasons of interest for this paper, and for many practical situations, are:

1. Errors in aiming, and
2. Dispersion of fire about the *actual* aim point as distinct from the *intended* aim point.

At first let us assume that all observations of impact should be given equal weight; this is to say that for each shot the sum of the variances, because of dispersion and errors in aiming, is the same. Further, we shall restrict ourselves to consideration of a miss in only one direction.

Consider shot No. 1, and suppose it is seen to hit at a point x_1 in the horizontal line, with 0 being both the bull's-eye and the point at which x is zero. The shot may have missed 0 by the amount x_1 for either of the reasons listed above or for a combination of both. Although it may not be the correct estimate, the best estimate of the actual aim point is x_1; so the best course to take to correct the aim is to shift the aim point by $-x_1$ before firing again.

Suppose shot No. 2 is seen to hit at x_2. If we had not shifted the aim point, it would have hit at $x_1 + x_2$. Further, if we had not shifted the aim point, the best estimate of the actual aim point would have been the mean of the observed impact points, i.e., the mean of x_1 and $(x_1 + x_2)$, i.e., $[x_1 + (x_1 + x_2)]/2 = x_1 + (x_2/2)$. This tells us that if we had not shifted the aim after shot No. 1, the best thing to have done after shot No. 2 would have been to shift the aim point by $-[x_1 + (x_2/2)]$. Since, in fact, we have already shifted by $-x_1$ after shot No. 1, all that is left to do after shot No. 2 is to shift by $-x_2/2$, i.e. by the negative of half the observed miss distance on shot No. 2.

Suppose that after shifting the aim by $-x_1$ after shot No. 1 and $-(x_2/2)$ after shot No. 2, shot No. 3 is seen to hit at x_3. We can see that if we had not shifted aim points:

Shot No. 1 would have been reported to hit at x_1;

Shot No. 2 would have been reported to hit at $x_1 + x_2$; and

Shot No. 3 would have been reported to hit at $x_1 + x_2 + x_3$.

The mean position of these three impacts would have been one-third of the sum, i.e., $x_1 + (x_2/2) + (x_3/3)$. So the best aim point shift after shot No. 3, supposing no earlier shifts, would be $-[x_1 + (x_2/2) + (x_3/3)]$. But if a shift of $-x_1$ took place after shot No. 1 and $-(x_2/2)$ after shot No. 2, the extra shift required after shot No. 3 is $-(x_3/3)$.

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[1] K. F. Gauss, *Theory of the Motion of the Heavenly Bodies Moving About the Sun in Conic Sections,* C. H. Davis (trans.) Dover Edition, 1963.

The above line of argument can be extended indefinitely to show that if a shift of aim point is to be made after each shot, it should be by the amount $-x_i/i$, where x_i is the observed miss after shot No. i. Since the shift made at each stage is optimum, the whole operation of shifting by $-x_i/i$ after the i^{th} shot, where i is 1, 2, 3, . . . , etc., is overall the best way of aiming at a target when the results of the fire can be observed after each shot. The course suggested is optimal for a horizontal shift. Similar action for a vertical shift optimizes the aim vertically.

Since a shift of x_i/i horizontally and a shift of y_i/i vertically is equivalent to a shift of one i^{th} of the radial error toward the bull's-eye, we can write a very simple correction rule which is valid for two or more dimensions.

The Rule—After the i^{th} shot, where i is successively 1, 2, 3, . . . , shift the aim toward the target by one i^{th} of the observed miss (that is, by $1/i$ times the reported error).

It is of interest to consider the errors involved in following the above rule.

Suppose that the dispersion associated with each shot has a variance of σ^2 and that the error in aiming each shot has a variance of σ_o^2. The variance of the reported position of a shot with respect to the actual aim point is thus $\sigma^2 + \sigma_o^2$ for each shot. When the rule is applied $n-1$ times, the correction applied amounts to shifting the aim point by the mean of the reported miss distances. The variance of this mean is $(\sigma^2 + \sigma_o^2)/(n-1)$, where $n \geq 2$.

The dispersion of the n^{th} shot has the single shot variance, namely σ^2. The variance in the position of the n^{th} shot with respect to the bull's-eye is the sum of the variance of the mean and the single shot variance, or $\sigma^2 + (\sigma^2 + \sigma_o^2)/(n-1)$. Thus, as n increases, the variance in the position of the n^{th} shot about the bull's-eye tends to σ^2. This is the variance due to the dispersion of the n^{th} shot. It is possible to reduce it only by improving the firing device.

Hitherto we have assumed that the variances of observations are constant from round to round. If the variances of observations do vary from round to round, as with some devices used for observing artillery fire, then the preceding rule can be generalized: After the n^{th} shot, shift on the x-axis towards the target by

$$\frac{1}{(\sigma^2 + \sigma_n^2) \sum_1^n \frac{1}{\sigma^2 + \sigma_i^2}} x_n ,$$

where σ is the standard deviation of the fall of shot along the x-axis, σ_i is the standard deviation of the observation of the i^{th} impact, and x_n is the n^{th} reported miss distance. This rule is derived in a manner similar to that for the previous rule and is identical to it if σ_i is constant. A similar rule obtains for adjustment along another axis.

The distance between the target and the n^{th} impact along the horizontal axis is a random variable with mean zero and variance

$$\frac{1}{\sum_1^{n-1} \frac{1}{\sigma^2 + \sigma_i^2}} + \sigma^2 .$$

The distances from the target along other axes obey the same statistical law, *mutatis mutandi.*

Conclusions

The optimum way to shift fire to take account of successive observed misses in aimed fire against a target has been given. The rule, as given, is a simple one. Since the rewards of using the optimum aim shifting procedure may be very great, it is surprising that no earlier mention of it has been found. It may very well be that good marksmen intuitively follow a rule close to that suggested.

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