SATELLITE GEODESY

W. H. Guier

Geodesy is the science of determining the size and shape of the earth and the gravity field associated with the earth. The motions of artificial near-earth satellites are extremely sensitive to the earth's gravity field, of course, and it was inevitable that their advent would give new impetus to geodesy. However, what was not recognized initially was the major degree to which satellites would change our knowledge of geodesy and the extent to which they would become one of the principal tools of the modern geodesist.

Probably the primary reason for not immediately recognizing the usefulness of satellites is that at the time of the first satellite launchings it was believed that the size, shape, and gravity field of the earth were sufficiently well known already and that no surprises would come from the study of satellite motion. However, it was not long before such surprises did appear. First, it was found almost immediately that the bulge of the earth's equator is smaller than previously believed.[1] In 1959 O'Keefe, Eckels, and Squires noted that there is a pronounced asymmetry between the northern and southern hemispheres, giving a "pear" shape to the earth.[2] In 1961 R. R. Newton and I. Izsak independently found that there is a significant departure of the earth from rotational symmetry, giving an elliptic shape to the earth's equator.[3,4] Since then new findings have appeared with such rapidity that one might say that modern geodesy is no older than the launching of the first satellites.

It is now generally agreed that the use of artificial near-earth satellites for geodetic investigations is an established discipline within the science of geodesy. There now exist huge quantities of experimental satellite data together with specialized methods for analyzing them; and large numbers of competent investigators are actively engaged in the field. Unfortunately the field of satellite geodesy has two additional attributes of an active discipline. First, it is impossible for any one investigator to keep fully informed on all current activities, and it certainly is difficult to report on more than a fraction of the current progress within any single paper. Second, progress is so rapid that any report is out of date by the time it appears in print. The present paper is limited to reporting the most recent progress in the determination, by means of satellite doppler data, of the earth's gravity field, and is "recent" only as of December 1, 1964.

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[1] J. A. O'Keefe, "Zonal Harmonics of the Earth's Gravitational Field and the Basic Hypothesis of Geodesy," *J. Geophys. Res.*, **64**, 1958, 2389–2392.

[2] J. A. O'Keefe, Ann Eckels, and R. K. Squires, "The Gravitational Field of the Earth," *Astron. J.*, **64**, 1959, 245–253.

[3] R. R. Newton, "Ellipticity of the Equator Deduced from the Motion of Transit 4A," *J. Geophys. Res.*, **67**, 1962, 415–416.

[4] I. G. Izsak, "A Determination of the Ellipticity of the Earth's Equator from the Motion of Two Satellites," *Astron. J.*, **66**, 1961, 226–229.

Artificial near-earth satellites are prolific sources of geodetic information. This paper is a brief review both of the types of satellite data that have been used extensively for geodetic determinations, and of basic approaches to inferring geodetic parameters from the data. In particular, the use of range rate (radio doppler) to determine values for the non-zonal coefficients through eighth order and degree are given. The most recent activity in the use of satellite data is to determine coefficients of the geopotential for degree and order of 13 and 14 through the effects of near-resonance between the satellite motion and these components of the geopotential. Preliminary values are given for degree and order 13.

The Geopotential and Its Orbital Perturbations

The gravitational force field of the earth is usually described through the expansion of the gravitational potential energy of the earth and does not include the "centrifugal" potential caused by the earth's spin relative to inertial space. This potential energy function, called the geopotential, is given in Eq. (1) where the usual expansion in Legendre spherical harmonics has been made.

$$U(r, \phi, \lambda) = \frac{K}{r}\left[1 + \sum_{n=2}^{\infty} J_n\left(\frac{R_o}{r}\right)^n P_n(\sin\lambda) + \sum_{n=2}^{\infty}\left(\frac{R_o}{r}\right)^n \cdot \sum_{m=1}^{n} J_n^m P_n^m(\sin\phi)\cos m(\lambda - \lambda_n^m)\right], \quad (1)$$

where r is geocentric radius; ϕ is geocentric earth-fixed latitude, λ is geocentric earth-fixed longitude, R_o is mean equatorial radius of earth, J_n is zonal harmonic coefficient of degree n, and J_n^m, λ_n^m are non-zonal harmonic coefficients of degree n and order m.

This expansion has the earth's center of mass as its origin, hence the absence of $n = 1$ in the expansion. The sum has been separated into those terms that describe rotationally symmetric contributions to the geopotential (that is, rotationally symmetric about the earth's spin axis) and those terms that depend on longitude and therefore are not rotationally symmetric.

Satellite orbits remain more or less fixed in inertial space, of course, with the earth rotating under the orbit. It can be seen that those terms that are rotationally symmetric exhibit no special effect on satellite motion because of the earth's rotation; those, however, that are dependent upon longitude produce periodic forces on the satellite whose fundamental period is the time for one complete revolution of the earth under the satellite orbit. For this reason the effects on satellite motion of the rotationally symmetric terms are fundamentally different from the effects of those that are not rotationally symmetric; different data requirements and analysis techniques are used to determine the two different types of coefficients. Historically, the terms that are rotationally symmetric are called zonal spherical harmonics, and the non-rotationally symmetric ones are called non-zonal spherical harmonics. Those terms for which $n = m$ have a special designation and are called sectorial harmonics. Those for which $0 \neq m \neq n$ are often called tesseral harmonics. In terms of these harmonic coefficients, the equatorial bulge is defined principally by the value of the even zonal coefficient J_2. The major asymmetry between northern and southern hemispheres—the pear-shape term—is given by the value of the odd zonal J_3 term. The discovery of the elliptic equator came from finding for the first time a non-negligible value for the sectorial harmonic coefficient J_2^2.

Considering first the effects of the even zonal harmonics, a further separation into those of even degree and of odd degree is necessary. Equations (2a) and (2b) indicate to the crudest approximation the principal effects of the even zonal harmonics:

$$\frac{d\Omega}{dt} = -\frac{3}{2} J_2 \left(\frac{R_o}{a}\right)^2 \bar{n} \cos i + 0[(J_2)^2] + 0(J_4), \qquad (2a)$$

and

$$\frac{d\omega}{dt} = \frac{3}{4} J_2 \frac{R_o}{a} (4 - 5 \sin^2 i)\, \bar{n} + 0[(J_2)^2] + 0(J_4)\,, \qquad (2b)$$

where Ω is ascending node, ω is argument of perigee (angle from node to perigee in orbital plane), a is semi-major axis, i is inclination, and $\bar{n}$ is mean motion (2π/satellite period). In these equations and those to follow, $0(x)$ indicates that terms of the order of x are neglected where x is small. Note that these time derivatives for the node and argument of perigee are of nearly constant value and produce precession or secular effects in these two angles. These two precessions are well known, and they describe an orbit whose perigee continually rotates within the orbital plane and whose ascending node (for orbits whose inclinations are less than 90°) precesses westward. Of all the even zonals, J_2 is the largest by about a factor of 10^3. It is this harmonic that represents the principal contribution to the equatorial bulge of the earth arising from the centrifugal force of the earth's rotation. Since J_2 is so large, the equations of motion for a satellite must be derived to second order in J_2 before accurate comparisons with experimental data can be made. Typically, the even zonals produce one complete revolution of the ascending node around the equator and one complete rotation of perigee in from one to three months.

Equations (3a) and (3b) indicate to an equally crude approximation the principal effects of the odd zonal harmonics:

$$\epsilon = \epsilon_o - \frac{1}{2}\frac{J_3}{J_2}\frac{R_o}{a} \sin i \sin\omega + 0\left(\frac{J_5}{J_2}\right); \qquad (3a)$$

and

$$\omega = \omega_0 - \frac{1}{2}\frac{J_3}{J_2}\frac{R_o}{a}\frac{\sin i}{\epsilon} \cos\omega + 0\left(\frac{J_5}{\epsilon J_2}\right), \qquad (3b)$$

where a is semi-major axis, ϵ is eccentricity, i is inclination, and ω is argument of perigee. Here the principal effects are an oscillation in the value of the eccentricity and in the argument of perigee. The period of the oscillation is the time of one full revolution of perigee. Typically the odd zonal harmonics produce amplitudes of the oscillation in the altitude of perigee of several tens of miles.

It can be seen that the effects of the zonal harmonics on satellite motion are very large and easily observed. In addition when these effects are examined in detail it is found that each degree zonal has its own characteristic dependence upon the orbit inclination. Consequently the most useful form of experimental data for evaluating the zonal harmonic coefficients are tabulations of experimentally determined values of the orbital parameters over very long periods of time at many different inclinations. In principle, a fit of the theoretically calculated time-dependence of the orbit parameters as functions of the zonal harmonic coefficients to the tabulated experimental parameters provides a determination of values for the coefficients. The procedure is not as straightforward as one might think from this simplified explanation. The theory also must include accurately the effects of atmospheric drag, solar radiation pressure, sun and moon gravitational effects, and, before accurate determinations can be made, a proper averaging of the errors caused by errors in the non-zonal harmonic coefficients. Consequently, such investigations become a real tour de force in perturbation calculations of satellite orbits.

The effects of the non-zonal harmonics on satellite motion are fundamentally different from the effects of the zonal harmonics just described. This basically is because the non-zonal harmonics, being longitude-dependent, give rise to satellite forces that are periodic, with a fundamental period equal to the time for the earth to make one complete revolution under the satellite orbit. This period is the sidereal day less the distance that the node has precessed westward, stated in units of time. Depending upon the order, m, of the non-zonal, various harmonics of the basic period can appear in the forces. The principal angular frequencies that appear are summarized in Eqs. (4a) and (4b):

$$2\pi f^m_{n,o} = m\,[\omega_E - \dot{\Omega}]\,, \qquad (4a)$$

and

$$2\pi f^m_{n,\pm} = \bar{n} \pm m\,[\omega_E - \dot{\Omega}]\,,$$

where the potential term of degree n and order m is:

$$U^m_n = \frac{K}{r} J^m_n \left(\frac{R_o}{r}\right)^n P^m_n(\sin\phi) \cos m(\lambda - \lambda^m_n)\,, \qquad (4b)$$

and where f is frequency of the non-zonal harmonic motion, $\bar{n}$ is satellite mean motion (radians/sec), and ω_E is angular rotation rate of earth (radians/sec).

Forces with these angular frequencies cause deviations in the satellite position, which are also periodic with periods of less than one day. Furthermore, with such short periods the amplitudes of the deviations are small because the forces change sign before the satellite can accelerate to large deviations in position. Typical amplitudes for these oscillations in position are a few tenths of one mile. Consequently, unlike the data required for the zonal coefficients, the experimental data most useful for evaluating the non-zonal coefficients are those that are extremely dense in time, preferably at least one set of data for each revolution of the satellite.

Data that are this dense in time must be supplied by a world-wide tracking network that has simultaneously existing tracking stations more or less uniformly distributed throughout the world. Furthermore, the tracking network must supply extremely accurate data to permit accurate measurement of such small amplitude oscillations. Finally, a serious difficulty is presented if large gaps in the distribution of data occur synchronously with time of day because of the need to follow accurately many periods of the oscillations. This last requirement cannot be met by optical tracking networks; it requires a radio tracking system.

The TRANET Tracking System

At least one tracking system that meets these requirements does now exist. It is known as the TRANET Doppler Tracking System, supported by the U. S. Navy.[5] Since this network is so fundamental to accurate determinations of the non-zonal harmonic coefficients, it will now be described, along with the preliminary processing and archiving of the data used to determine the coefficients.

Figure 1 shows the number and general distribution of doppler tracking sites that currently exist. One additional station expected to be added to the system during 1965 will be located at McMurdo Sound, Antarctica. This is an impressive distribution of stations and currently supplies upwards of one hundred sets of doppler data per satellite per day. It can be seen that there are considerably more stations in the northern than

[5] R. R. Newton, *Description of the Doppler Tracking System TRANET,* TG-571, The Johns Hopkins University, Applied Physics Laboratory, May 1963.

Fig. 1—Number and general distribution of doppler tracking sites currently in use and immediately planned in the Navy satellite tracking system, TRANET.

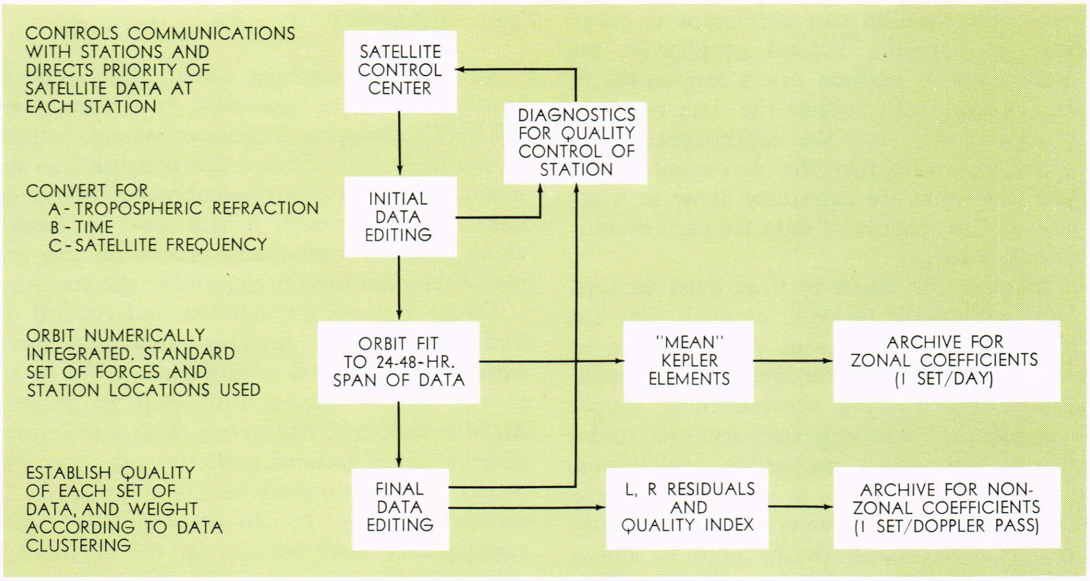

Fig. 2—Diagram showing the flow of doppler data from the Satellite Control Center to the data archives.

in the southern hemisphere and a dense cluster within the 48 continental States. While such clustering of the stations is extremely valuable when making cross checks for quality control of the data, it usually is necessary to reduce the weights attached to the data from some stations to provide a more uniform statistical control when determining non-zonal coefficients.

Each station is equipped to receive simultaneously at least two doppler frequencies from each satellite and to combine them in analog circuitry in such a way that the principal effect of ionospheric refraction is eliminated.[6] In addition, surface meteorological data are transmitted with each set of "vacuum" doppler data to the Satellite Control Center at APL, to aid in making accurate a priori corrections for tropospheric refraction.[7]

Figure 2 indicates the flow of data from the Satellite Control Center to the data archives, two of which are maintained. The first is a set of mean Kepler orbital elements. Typically, these elements are averages of the osculating elements over a 24-hr time period and are used for improving values of the zonal harmonics coefficients. To date they have been used by R. R. Newton for improving values for the odd zonal harmonics.[8] The second archive is a collection of all passes of all stations of each satellite that have been judged to be of sufficient quality to be used in evaluating the non-zonal harmonic coefficients. An example of the factors included in the data quality index are indices for the magnetic and solar activity for the day of the pass to aid in estimating the accuracy of the analog correction for first order ionospheric refraction.

Figure 3 indicates the quality of doppler data that are obtained, showing the difference between the theoretical doppler shift and experimental doppler data. This sample is for an unusually low elevation in order to emphasize the effect of tropospheric refraction and to indicate a rough upper bound for the noise level that is typically received. The noise level is roughly 2 parts in 10^{10}. At this low an elevation the data are relatively insensitive to errors in the satellite trajectory and therefore would never be used for geodetic research.

Typically, each set of doppler data comprises between 200 and 400 individual data points, and consequently, some method of aggregating the data is needed for archiving. A method that is efficient and well adapted to studying the effect of the non-zonal harmonics has been found. The aggregation parameters are called the along-track (L) and slant-range (R) residuals. The validity of these parameters is based on the following theorem:[9]

[6] W. H. Guier, "Ionospheric Contributions to the Doppler Shift at VHF from Near-Earth Satellites," *Proc. IRE,* **49,** Nov. 1961, 1680–1681.

[7] H. S. Hopfield, "The Effect of Tropospheric Refraction on the Doppler Shift of a Satellite Signal," *J. Geophys. Res.,* **68,** Sept. 1963, 5157–5168.

[8] W. H. Guier and R. R. Newton, *The Earth's Gravity Field Deduced from the Doppler Tracking of Five Satellites,* TG-634, The Johns Hopkins University, Applied Physics Laboratory, Dec. 1964.

[9] W. H. Guier, *Studies on Doppler Residuals—I: Dependence on Satellite Orbit Error and Station Position Error,* TG-503. The Johns Hopkins University, Applied Physics Laboratory, June 1963.

To first order, the range and range rate residuals can be reduced to their experimental noise level by appropriately adjusting the satellite position at time of closest approach, t_c, in the plane defined by the slant range vector and satellite velocity vector, also evaluated at t_c.

The geometry appropriate to this theorem is shown in Fig. 4. The theorem is proved by noting that when the error in the satellite trajectory is small the actual and estimated trajectories are nearly parallel. Consequently, the relative geometry between satellite and station is approximately preserved during the whole time of the satellite pass by so adjusting the satellite position that the slant range and time of closest approach agree with the experimental values.

These are convenient parameters to use for archiving because they indicate directly experimental values for the relative position error between satellite and station. For example, the error level of the data for geodetic purposes can be estimated by comparing the *L*- and *R*-residuals from co-located doppler stations where errors in orbital trajectory, refraction, and station location do not affect the results. Such comparisons are frequently made as checks on the quality of the satellite and station instrumentation; characteristically, they show an instrument precision of between 10 and 20 meters. Absolute accuracy of data cannot be so easily established because errors in the knowledge of the geopotential produce satellite position errors much larger than this. It is believed generally to be between 10 and 30 meters and is caused principally by inadequate accounting for ionospheric and tropospheric refraction errors.[5]

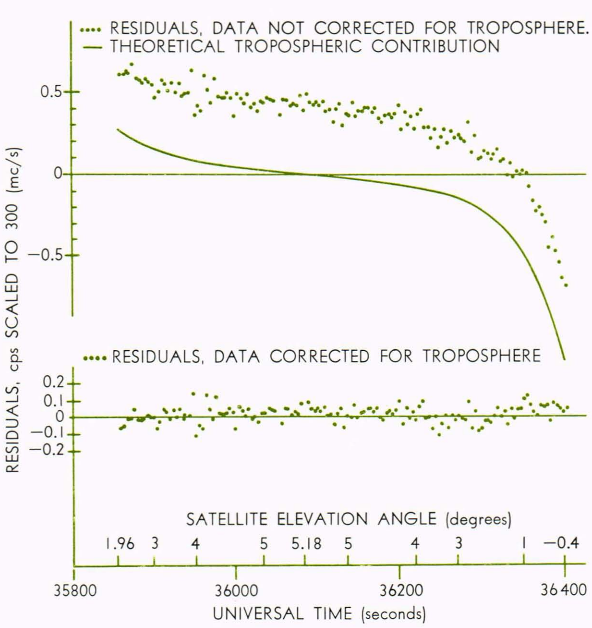

Fig. 3—Doppler residuals with and without tropospheric correction, showing the tropospheric contribution to the experimental doppler data.

Inference of the Non-Zonal Coefficients

Having discussed the tracking network and the experimental data that are obtained for determining the non-zonal coefficients, the method for evaluating the latter can be quickly stated. The satellite perturbing equations of motion for any non-zonal harmonic can be written in position coordinates that are closely connected to the *L*- and *R*-experimental residuals. These changes in position, caused by a change in the value of a non-zonal coefficient, are defined in Eqs. (5a) through (5e), along with the expressions to first order for the *L*- and *R*-residuals. It can be seen that the perturbed along-track coordinate, $L_n^m(t)$, is essentially the same as the *L*-residual when evaluated at t_c. The perturbed cross-plane coordinate, $C_n^m(t)$, and the altitude coordinate, $H_n^m(t)$, are combined according to the elevation of the pass to give the slant range *R*-residual.

$L_n^m(t)$ = change in satellite along-track position for change in (n,m) non-zonal harmonic; (5a)

$H_n^m(t)$ = change in satellite radius for change in (n,m) non-zonal harmonic; (5b)

$C_n^m(t)$ = change in satellite position normal to orbital plane for change in non-zonal harmonic; (5c)

Contribution to *L*-residual = $L_n^m(t_c)$ + Second Order, (5d)

and

Contribution to *R*-residual = $C_n^m(t_c) \cos e + H_n^m(t_c) \sin e$ + Second Order, (5e)

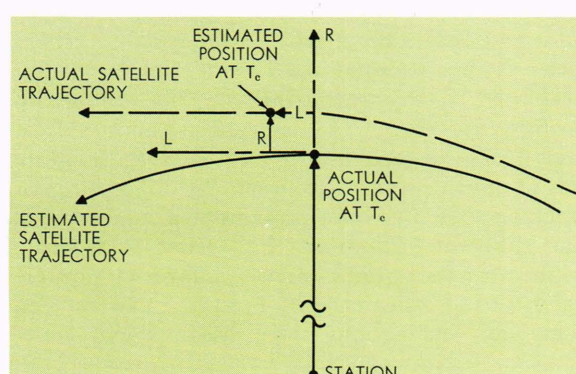

Fig. 4—The along-track and slant-range residuals associated with the theorem discussed in the text.

TABLE I

DOPPLER PASSES USED FOR DETERMINING NON-ZONAL COEFFICIENTS THROUGH J_8^8, λ_8^8 AND J_{13}^{13}, λ_{13}^{13}

Satellite	*Mean Anomalistic Period (minutes)*	*Mean Inclination (degrees)*	*Number of Passes*	*Number of Different Station Sites*	*Mean Residuals (meters)* ($\sqrt{L^2 + R^2}$)
1961 $\alpha\eta$ 1	105.8	32.43	214	9	72
1962 $\beta\mu$ 1	107.9	51.13	561	14	66
1961 o1	103.9	66.80	237	11	89
1963 38C	107.3	89.91	301	12	79
1963 49B	107.1	89.96	349	16	79
		Totals	1662	21	77 *m* = 0.042 *nm*

where t_c is the time of closest approach, and e is the elevation of the satellite above the station's horizon at t_c.

The detailed results of the perturbation theory for the non-zonal coefficients are too lengthy to present here.[10] Each non-zonal harmonic produces its own unique dependence of the three coordinates upon time and satellite inclination. Consequently, in principle, with sufficient amounts of experimental L- and R-residuals from a sufficient number of satellites at different inclinations, values for all non-zonal harmonics that contribute more than about 10 meters of position error can be determined. The practical problems are those of obtaining enough satellite orbit inclinations and developing computer programs that can perform the huge job of least-squares fitting of the coefficients to the L- and R-residuals.

Table I presents a summary of the data that were used for the most recent determination of the non-zonal coefficients through $n = m = 8$.[8] (The two coefficients for the sectorial, $n = m = 13$, will be discussed later.) Recognizing that each non-zonal requires two coefficients to specify it (amplitude and longitude of its maximum value), there were 70 geopotential coefficients simultaneously evaluated in this computation. This number of coefficients requires a considerable amount of data, and it can be seen that a total of 1662 separate sets of data were used from a total of 21 different tracking sites. One can prove that a minimum of four different inclinations are needed to separate definitively the various degrees, n, for each possible value of the order m, $m \leq 8$, and it can be seen that four inclinations were used. These cover inclinations between approximately 30° and 90°, are in 20° increments.

Of all the data that had been archived at the time of this determination, the data indicated in Table I were chosen as being of the highest quality, the most uniformly distributed in time, and the most uniformly distributed over the surface of the earth. Because of the huge quantity of data available, a special data-search-select-and-retrieval program was written to reduce to a minimum the human time involved in the selection.

To give some appreciation for the magnitude of this determination, it should be pointed out that not only must a least-squares fit to the geopotential coefficients be made, but also the station locations and orbit parameters must be adjusted to values that minimize the L- and R-residuals before the most accurate values for the non-zonal geopotential coefficients can be obtained. This computation, overall, required approximately 50 hours of IBM 7094 computer time once the final set of data was selected. It can be seen that the overall position residuals, that is, rms of $[L^2 + R^2]^{1/2}$, is 77 meters and provides an indication of the level of geopotential errors that remain.

The assumption that this level of the residuals is still caused principally by remaining geopotential errors is based on a detailed examination of the character of the time-dependence of the residuals. Such an examination indicates that the residuals exhibit a characteristic pattern in time-dependence that is repetitive day after day and is strongly correlated with the earth-fixed longitude of the ascending node and, for some of the satellites, is correlated with the latitude of the satellite. Making the assumption that these correlated errors are the result of errors in the geopotential, the remaining errors exhibit essentially a random character with a magnitude of between 10 and 30 meters.

[10] W. H. Guier and S. M. Yionoulis, *A Perturbation Theory for Satellite Along-Track, Altitude, and Cross-Track*, TG-635, The Johns Hopkins University, Applied Physics Laboratory (in preparation).

Resonant Harmonics

Such examinations of the character of the errors during the computation process for the non-zonal coefficients has led to the most recent investigations. Figure 5 is a plot of the along-track residual L at one point during the evaluation of the geopotential coefficients. It can be seen that there is a pronounced periodic oscillation in these L-residuals with an amplitude of about 0.08 nautical mile and a period of about 60 hr. The explanation for the cause of this oscillation was first given by S. Yionoulis (APL):[11] the non-zonal harmonics of order $m = 13$ produce oscillations almost exactly equal to the orbital period for this satellite, and the 60-hr period of the L-residuals is the beat between the $m = 13$ non-zonals and the satellite orbital period. Considering the most probable value for the degree n that should be associated with $m = 13$, n must be odd,[11] and the largest influence of all the odd degree harmonics of order 13 for a polar satellite is the sectorial harmonic $n = m = 13$. The two coefficients for this sectorial were then evaluated along with the other 68 lower harmonic coefficients to produce the final residuals shown in Table I.

Figure 6 shows the result of including the resonant sectorial harmonic (13,13). While a marked decrease in the L-residuals has occurred, there still exists a very small component with a 60-hr period and, now that the 60-hr component has been reduced, a very small component with a period of about 40 hr can be noticed. Also a close examination of the detailed structure of the residuals reveals that they exhibit a strong component at a very short period—somewhat less than 2 hr. This short period does not appear to be the satellite orbital period but is at a period of 1/14 of a sidereal day. This suggests that a non-zonal harmonic of order $m = 14$ may not be negligible, and indeed a beat between the period of the $m = 14$ non-zonals with the orbit period yields a 40-hr period for this particular satellite. Thus it is now clear that resonating non-zonal harmonics of degree $m = 14$ will have to be included. In fact it appears now that coefficients for the resonant harmonics of order $m = 13$, 14 and degree $n = 13$, 14, 15 will have to be evaluated before the L- and R-residuals can be reduced significantly below their current values.

The study of these resonating non-zonals is our principal effort at present. The perturbation theory for resonating forces is much more difficult than for low values of m, and further evaluation of the resonant coefficients is awaiting completion of the code to compute their perturbing effect numerically. Further refinement of the lower order non-zonal coefficients is probably not possible

[11] S. M. Yionoulis, *Resonant Geodesy,* TG-633, The Johns Hopkins University, Applied Physics Laboratory, Dec. 1964.

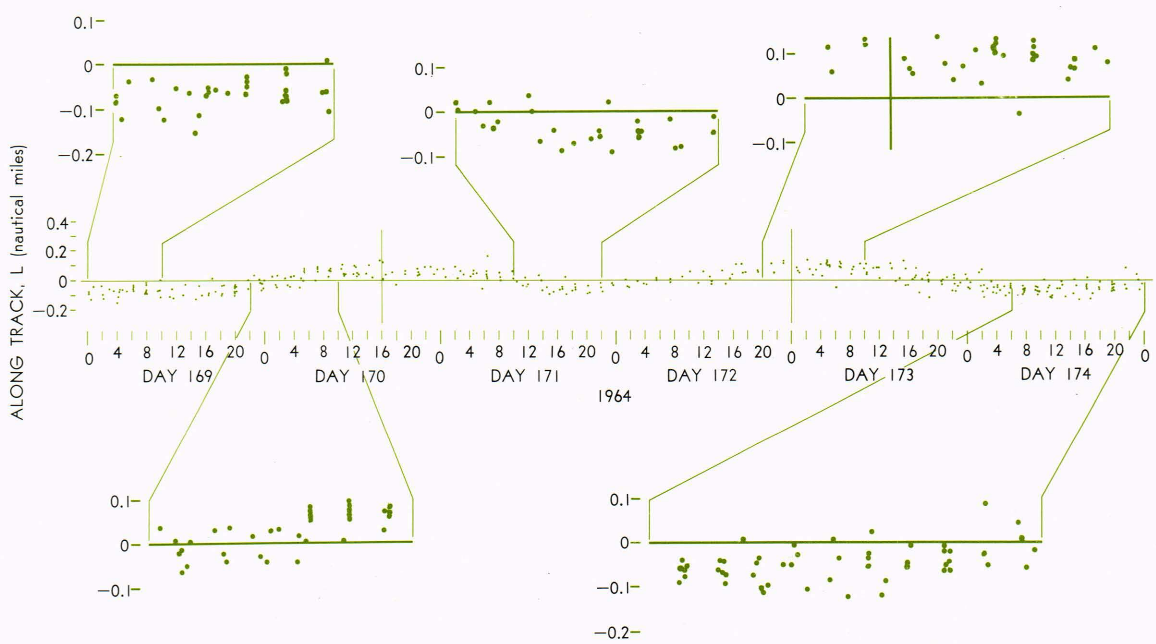

Fig. 5—Plot of the along-track residual-*L* from data taken by Satellite 1963 49B, showing the (13,13) resonance effect.

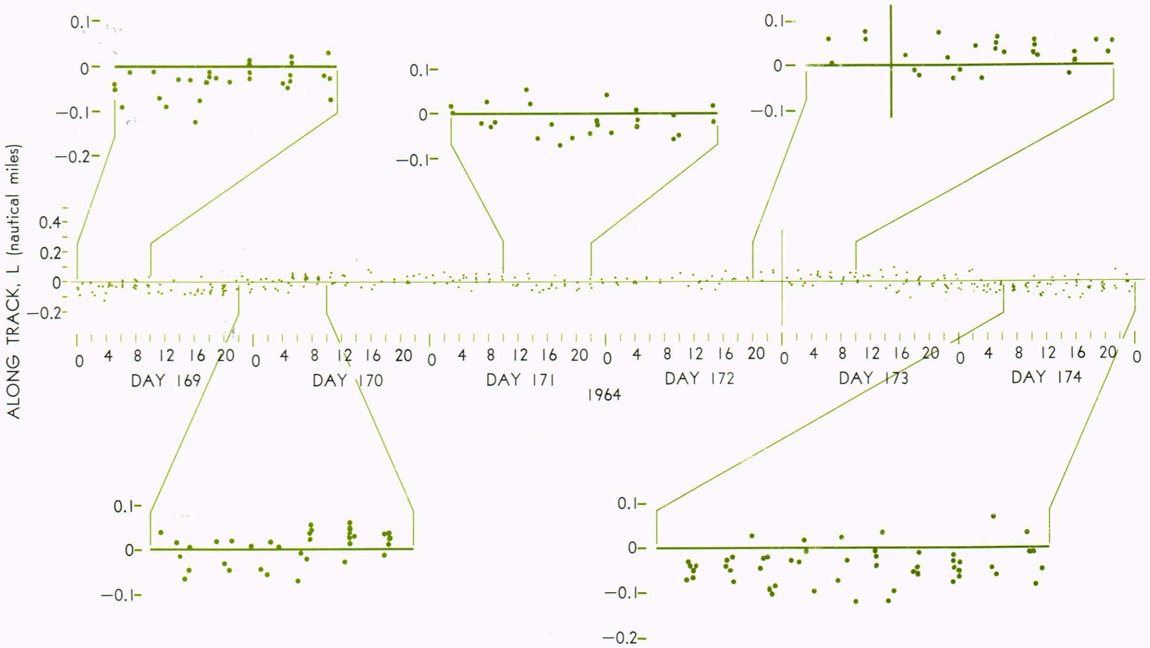

Fig. 6—Plot of the along-track residual-L from data taken by Satellite 1963 49B, where the (13,13) resonance has been accounted for.

until the very large effects of the resonating harmonics are accounted for.

Latest Results

Table II presents the results for the latest evaluation of the non-zonal harmonic coefficients. It includes all harmonics through the eighth sectorial, and preliminary values for the thirteenth sectorial These values are probably the most accurate set currently available. Table III presents values for the zonal harmonic coefficients; these are believed to be the most recent and complete. King-Hele[12] is the author of the values for the even-zonals, and R. R. Newton is the author of the odd-zonals.[8]

Tabulated values for this many coefficients provide little understanding of their overall effect on the geopotential. For this reason it has become standard practice to display models of the geopotential as contour maps of geoidal height. To the accuracy required here the geoidal height can be considered to be the difference in radius between that equipotential surface that coincides with mean sea level and a spheroid (ellipse of revolution about the spin axis). Figure 7 shows geoidal heights for this set of harmonics. The semi-major and minor axes of the spheroid have been adjusted to yield correct values for J_2 and J_4.

It is hopeless to attempt a valid estimate of the probable error for each of the harmonic coefficients, since the predominant error probably results from the constraining of higher degree harmonics to zero. However, there are indirect methods for checking the overall accuracy of the resulting geopotential. First, one can certainly use such a model of the geopotential for satellite tracking and to check the accuracy with which satellite positions can be predicted. As noted previously the current level of tracking error is about 80 meters, with the detailed character of the errors suggesting that the major contributor to these errors is the model of the geopotential itself. By using this geopotential, a few sets of optical data have been compared with the computed satellite position, and the optical-data residuals indicate an error level of about 100 meters. The difference between this optical error level and the doppler-tracking L-R residuals of 80 meters is believed to be caused by incompatible positions for the optical cameras and doppler tracking stations. Optical comparisons, after small adjustments have been made in the relative positions of camera and doppler stations, are currently in progress; preliminary results indicate an agreement to better than 80 meters.

[12] D. G. King-Hele, "Even Zonal Harmonics in the Earth's Gravitational Potential," *Nature,* **202,** June 6, 1964, p. 996.

TABLE II

VALUES FOR THE NON-ZONAL HARMONIC COEFFICIENTS

Sectorial (n/m)	Non-Zonal Harmonic Coefficients								
	1	2	3	4	5	6	7	8	
2	$J_2^1 \equiv 0$ $\lambda_2^1 = 0$	8.428 −13.35							
3	6.946 6.65	5.222 −14.56	4.421 18.66						
4	3.030 −141.95	2.587 23.36	3.591 0.16	1.203 34.49					
5	1.013 −50.60	2.024 −25.77	0.647 16.19	2.595 −37.99	3.140 −18.58				
6	0.515 90.12	1.141 −67.99	2.716 1.83	3.030 −30.27	2.748 −21.82	1.184 −14.41			
7	0.860 36.57	2.534 3.72	2.436 −9.23	0.749 44.95	1.072 −21.38	4.842 20.17	0.927 −8.36		
8	0.909 −161.89	0.594 −10.84	1.304 34.74	0.458 37.85	0.469 −0.36	3.884 15.31	1.071 −3.21	1.012 18.39	
13	—	—	—	—	—	—	—	—	0.490 10.36

(Entries are $J_n^m \times 10^6$ and λ_n^m in degrees. Legendre harmonics are normalized to $\pi_n^m = \sqrt{\frac{(n-m)!}{(n+m)!}}\,(1 - z^2)^{m/2} \frac{d^m}{dz^m} P_n\,(z)$).

Recently, W. Kaula has compared this geopotential with tracking results from the 24-hr satellite SYNCOM II.[13] This is an unusually interesting comparison. If the current errors in this geopotential are due principally to errors in high-degree non-zonals, including those that resonate with near-earth satellites, the low-degree non-zonals should be relatively more accurate than those of high degree. The SYNCOM II orbit has a sufficiently large radius that only low-degree harmonics contribute significantly to its motion. Consequently, Kaula's comparison provides a good check on the low-degree non-zonal harmonic coefficients, $n \lesssim 3$.

This comparison is shown in Table IV for the first two free drift periods of SYNCOM II. These low-order non-zonals are the principal contributors to an acceleration of the longitude of the satellite, and this drift acceleration is the quantity for which the comparison has been made. The principal contributors to the theoretical acceleration are the sectorials J_2^2 and J_3^3. This close agreement for such a radically different type of perturbation from that of the near-earth satellites (on which the coefficient evaluation was based) lends considerable confidence to the low-degree harmonics

[13] W. M. Kaula, Institute of Geophysics and Planetary Physics, University of California (private communication).

TABLE III

VALUES FOR THE ZONAL HARMONIC COEFFICIENTS

Degree (n)	Zonal Harmonic Coefficient ($J_n \times 10^6$)
2	−1082.70
3	+2.693
4	+1.40
5	+0.006
6	−0.37
7	+0.633
8	−0.07
9	−0.210
10	+0.50
11	No determination
12	−0.31

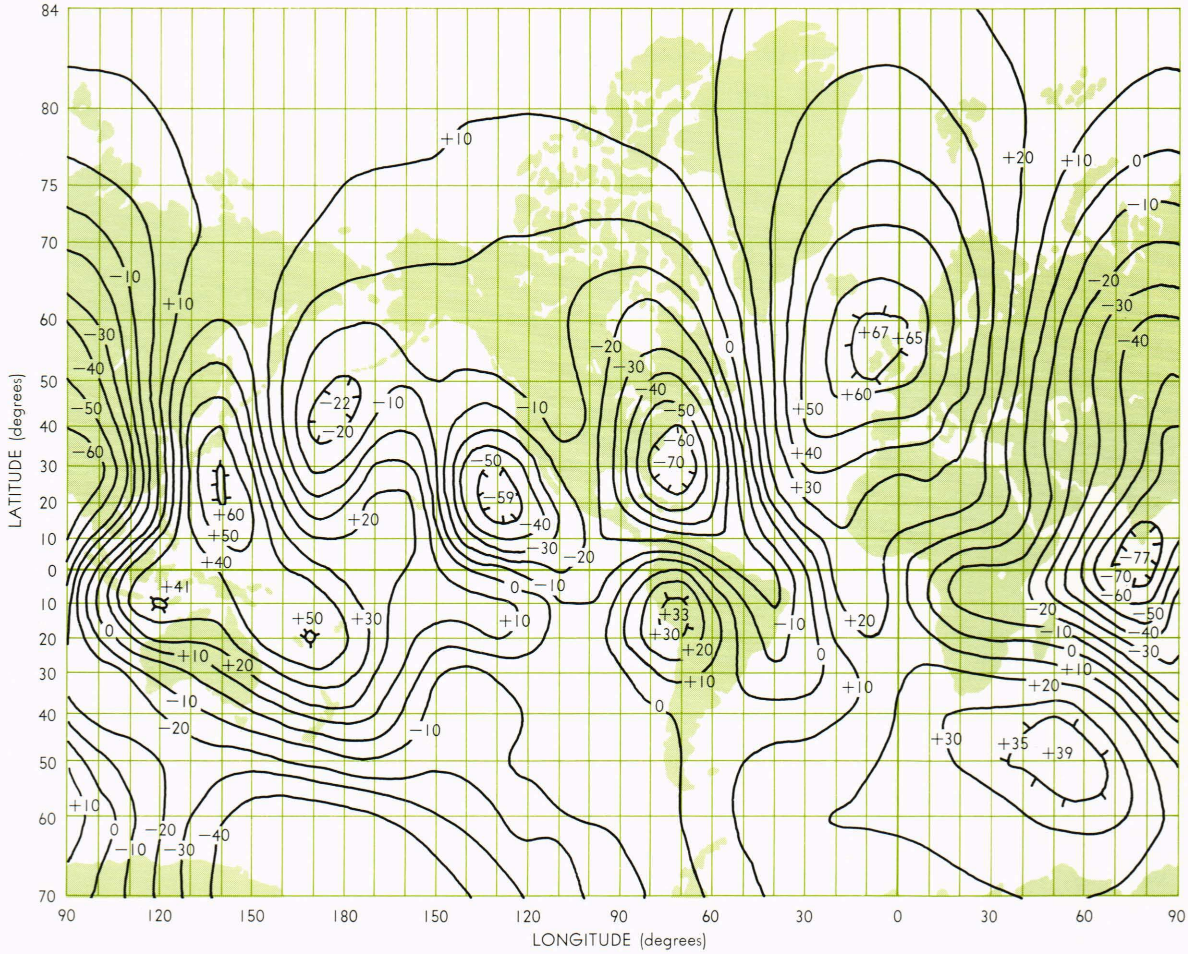

Fig. 7—Contour map of geoidal heights, in meters, for the geopotential coefficients given in Tables II and III.

TABLE IV

COMPARISON OF SATELLITE GEOID WITH SYNCOM II ACCELERATION IN LONGITUDE

*Experimental Mean Drift Acceleration**	*Theoretical Acceleration***	*Difference*
	(degree/day²)	
$(-1.27 \pm 0.02) \times 10^{-3}$ $(-1.32 \pm 0.02) \times 10^{-3}$	-1.26×10^{-3} -1.22×10^{-3}	$(-0.01 \pm 0.02) \times 10^{-3}$ $(-0.10 \pm 0.02) \times 10^{-3}$

* Average taken over about 81 days in each drift period.
** J_2^2, J_3^1, J_3^3, J_4^2, J_4^4 influences from W. Kaula (private communication).

being correct to at least 0.3×10^{-6}.

In summary, this geopotential is sufficiently accurate that near-earth satellites can be tracked to better than about 100 meters, and it can predict the drift acceleration of synchronous satellites to better than about $0.05° \times 10^{-3}$ /day². The errors in this geopotential, which are contributing most significantly to the tracking error of near-earth satellites, are those non-zonal harmonics that resonate with the satellite period, usually those for $n = 13$, 14, 15 and $m = 13$, 14. These tracking errors in turn are probably the major contributor to the errors in the evaluated coefficients of the lower order non-zonals, $n, m \lesssim 10$.