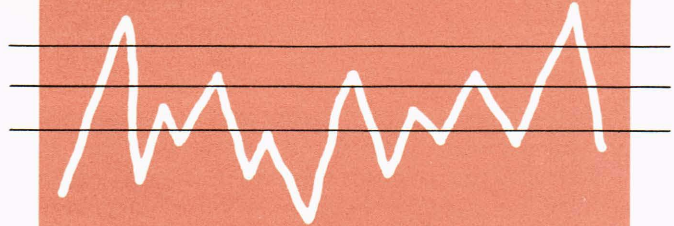


In an effort to extend the scope of existing data on radar detection probability, a way was found to express the pertinent relationships in such a way that accurate, extensive data could be computed on a high-speed digital computer. The present paper explains the physical significance of the data, the new mathematical solutions are described, and some representative graphs of the new data are given.

IMPROVED DATA for the Classical Determination of Radar Detection Range

L. F. Fehlner

Echoes from distant targets are characteristically so faint as to require highly sensitive receivers in radar systems. It follows naturally that the more sensitive the receiver, the greater the range that can be claimed for the radar. Unfortunately, however, echoes are not the only sources of low-intensity energy. Everyone is familiar with radio static and how it interferes with clear reception; the same kind of electrical noise interferes with the reception of radar echoes. Because of the inherent characteristics of noise, we cannot be absolutely sure that a detection made by a radar receiver is a target; it might be only noise. Thus, the problem becomes probabilistic, based on the nature of noise. Our decision criteria must distinguish between probable targets and probable noise.

Unavoidable noise is not the only thing that makes the specification of radar performance probabilistic. Perhaps targets are camouflaged by clutter from a sea or land background or by decoys; to make matters worse, perhaps the fluctuation in strength of the target echo resembles the fluctuation of noise or clutter. From an operational point of view, perhaps the radar fails during the period when targets are present; and perhaps targets are not encountered at the time and place predicted. All of these things affect the prediction of target detection and are treated under specialized headings such as reliability, maintainability, target fluctuation, clutter, signal processing, and tactical analysis. Our discussion, however, will consider only the classical problems of radar detection, i.e., the probability of detecting non-fluctuating and fluctuating targets when the only interference is electrical noise. The principal parameters that we need to consider are power transmitted, detection range, fluctuation characteristics, probability of target detection, and probability of false alarms. The relationships among these parameters are found by combining the mathematical descriptions of the physical processes and signal processing associated with the radar.

In 1947 and 1948, before high-speed digital computers were available, two papers by J. I. Marcum were addressed to the problem of the detection of echoes of non-fluctuating strength;[1] the test of time has proved them definitive. His treatment of the problem was statistical, and he was able to express the probability of detection as a function of signal-to-noise ratio under stated con-

[1] The work published by J. I. Marcum in 1947 and 1948, and extended in 1954 by P. Swerling, was published verbatim as "Studies of Target Detection by Pulsed Radar," in *Trans. I.R.E.*, ***IT-6***, April 1960.

ditions. The later of the two papers contains the basic statistical analysis. A substantial part of Marcum's text is devoted to the use of standardized functions and the description of mathematical approximations that made the calculations possible.

In 1954 P. Swerling extended Marcum's analysis to the case of a target with an echo of fluctuating strength.[1] His numerical results were obtained over a limited range of parameters, also through approximations.

The lasting contributions of these papers are the basic statistical analyses that resulted in rigorous analytic expressions for the probability of target detection in a noise background. The numerical data suffer on three counts, however: (1) lack of high-speed computers made it necessary to use approximations to the rigorous solutions; (2) laboriousness of hand computations made it necessary to limit the scope of the computed data; and (3) the effect of the approximations on the accuracy of the computations was not reported.

In connection with an effort to extend the scope of the data, a way was found to express the probability of detection in series form and without loss of any of the rigor of the former analyses. The summations required by these series were adaptable to high-speed digital calculation. Furthermore, the inaccuracy represented by the residue, caused by stopping the summation at any arbitrary term of the series, could be bounded. At this point, rather than merely increasing the scope of the data, it became feasible to recompute all the data presented by Marcum and Swerling. The new data were found to be correct, from the computational viewpoint, to six decimal digits.

In the present paper the applicability of the data is clarified, the new mathematical solutions are described, and some representative graphs of the new data are presented.

The Noise Problem

A pulse radar set is an electronic device that can alternately radiate and receive electromagnetic energy. Limitations are placed on the frequency bandpass of the receiver and also on the time of reception, with the result that any received energy is suspected of being an echo of the radiated energy. The receiver is designed to measure the smallest energy possible to maximize its performance since the maximum transmitted energy is always limited for a specific radar. As these receivers become more and more sensitive, their measurements become influenced by unwanted energy that previously had been unnoticed in the background.

Until recently background discrimination was principally a human task, in which trained observers watched optical displays that were attributable to the received energy and thermal noise generated internally. Through the use of very subtle target detection criteria and integration laws (that are not well understood), trained observers are very effective in detecting targets in the presence of background noise. Marcum's results, in theory, apply at least partially to target detection by people. For numerical results it would be necessary to interpret their detection criteria in terms of a signal-to-noise ratio, and also to know the observer's integration law.

Marcum's results apply directly to the performance of an automatic detection system that is based on a single threshold value of signal-to-noise ratio. In an automatic system, frequency discrimination is used to limit the noise energy, but it is not profitable to limit frequencies to less than those required to admit essentially all of the echo. Because noise contains these same frequencies superimposed in random phase and amplitude, noise over the bandwidth of the echo must be allowed to pass. The random superposition accounts for the large random fluctuations in the amplitude of noise.

The ratio of signal energy to noise energy is of fundamental concern in attempting to make radar detections. When signal energy is measured over the same time interval as noise energy, the energy ratio is identical to the power ratio. Since the times are usually the same, the equations that follow will be written in terms of power. It should be remembered, however, that in the general case the time interval for measuring signal and noise energy might not be the same; due account of this fact must therefore be taken.

The power of the target echo is

$$P_e = \left(\frac{P_t G_t}{4\pi R^2}\right)\left(\frac{\sigma}{4\pi R^2}\right)\left(\frac{G_R \lambda^2}{4\pi}\right)L, \qquad (1)$$

where P_e is power of the echo at the input to the receiver,
P_t is power transmitted,
G_t is gain of the transmitting antenna,
σ is the scattering cross section of the target,
G_R is gain of the receiving antenna,
λ is the carrier wavelength,
L is a factor to account for two-way losses due to such causes as propagation through the medium, antenna, beam shape, and plumbing, and
R is range to the target.

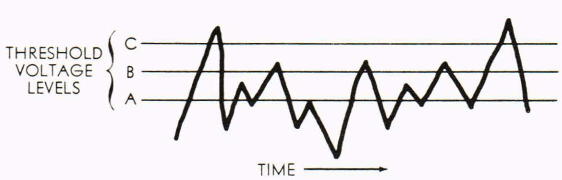
The first parenthetical term of Eq. (1) describes the radial distribution of power density from the transmitting (illuminating) antenna; the second term accounts for the power reradiated from the target; and the third defines the fraction of the reradiated power captured by the receiving antenna. The product of these terms defines a mean echo power that would be measured were it not for the large fluctuations in noise power. We could be reasonably confident that reports of targets by a properly adjusted radar would be real if we could establish a criterion for selecting a value of signal-to-noise ratio above which it is sufficiently improbable that the value is attributable to noise. Infrequent false alarms due to noise would have to be accepted as reports of real targets. Some detections would be missed because noise happens to be low during the echo. If false alarms prove bothersome, either the threshold value of the signal-to-noise ratio must be increased at the expense of missed detections or more radiated power, or more sophisticated sorting criteria must be instrumented. The latter alternative is beyond the scope of the present paper.

The problem of target detection by a continuous-wave (CW) radar can be treated in the same way if the observed sample of the continuous return from the target is interpreted as a pulse.

Statistical Solution

False Alarm Number—Marcum defined the complex relationship between a threshold value of signal-to-noise ratio and the probability that values in excess of the threshold will exist in the presence of both noise and echoes. These excesses are reported as target detections and, as mentioned above, include both false and real reports of targets. Also, unavoidably, targets sometimes will not be reported because the signal plus noise does not exceed the threshold. In statistics this decision criterion is called a Neyman-Pearson Observer.

The problem starts with noise. Assume that the voltage resulting from noise alone varies with time in the following way.



During the period shown in this illustration the noise would have exceeded threshold voltage level A seven times, level B five times, and level C only twice. Obviously, the higher the threshold the longer will be the average time between occasions when noise alone exceeds the threshold. This time is of considerable concern. If it is too short, we will be faced with too-frequent false alarms; if too long, excessive radiated energy will be required to achieve reasonable probabilities of target detection. Mathematically this time is defined as follows: false-alarm time is that during which the probability is P_0 that, in the absence of target echoes, there will not be a false alarm. For purposes of standardization P_0 is taken to be 0.5.

The probability that a detection is obtained each time there is an opportunity is given as P_N. False alarms are detections due to noise. The false-alarm number n' is the number of independent opportunities for a false alarm in the false-alarm time. Then, as standardized, the probability of not obtaining a false alarm in the absence of echoes is

$$P_0 = (1 - P_N)^{n'} = 0.5. \quad (2)$$

This equation expresses concisely the fact that the probability of an event not occurring in a number of independent trials is the product of the probabilities that it will not occur at each trial. When the number of opportunities for a false alarm, i.e. the false-alarm number, is very large, an approximation to Eq. (2) gives accurate values of the probability of a false alarm, namely,

$$P_N \approx \frac{1}{n'} \ln \frac{1}{P_0}. \quad (3)$$

The values of false-alarm probability in Table I apply when $P_0 = 0.5$.

TABLE I

PROBABILITY OF A FALSE ALARM IN THE ABSENCE OF TARGET ECHOES

False-Alarm Number	*False-Alarm Probability*
10^2	6.93×10^{-3}
10^3	6.93×10^{-4}
10^6	6.93×10^{-7}
10^8	6.93×10^{-9}
10^{10}	6.93×10^{-11}

The choice of false-alarm number associated with a particular radar depends on the function it performs. Search radars usually are designed to have very large false-alarm numbers, e.g. 10^6, to minimize the time wasted in reacting to false detections. On the other hand, track radars can use small false-alarm numbers because the tracking antenna is not unduly distracted by an occasional false detection during acquisition of the target;

and, during tracking, the distraction is completely negligible.

The number of opportunities for a false alarm may be calculated in terms of the pulse repetition frequency (PRF), the number of range gates per range sweep, and the method of signal processing. Common processing techniques are the coherent and incoherent integrations of pulses; coherency refers to the preservation of phase in the pulse-summing process. Range gates are switches that open and close at specified times so that a target echo, if any, can be admitted from a predetermined element of space referred to as a cell in Fig. 1.

Since a decision as to whether a target is present or not must be made each time a gate is open, noise pulses that enter can be interpreted as a target if their sum is strong enough. The number of pulses actually processed per unit time is the number of range gates per sweep multiplied by the PRF. If, prior to the decision, m pulses are integrated coherently, and the N of the resulting signals are integrated incoherently, the opportunities for a false alarm are reduced proportionally at the expense of reducing the number of cells that may be searched per unit time.

The false-alarm number is related to the false-alarm time as follows:

$$t_F = \frac{mNn'}{(PRF)G}, \qquad (4)$$

where t_F is the false-alarm time, m is the number of pulses integrated coherently, N is the number of pulses integrated incoherently, and G is the number of range gates per range sweep.

Assume that a radar is designed for a false-alarm number of 10^8; 10 pulses are integrated coherently, then 100 incoherently; the pulse repetition frequency is 10^4/sec, and the number of gates per sweep is 10^3. The false-alarm time is 2.78 hr; that is, in an average 2.78 hr of operation

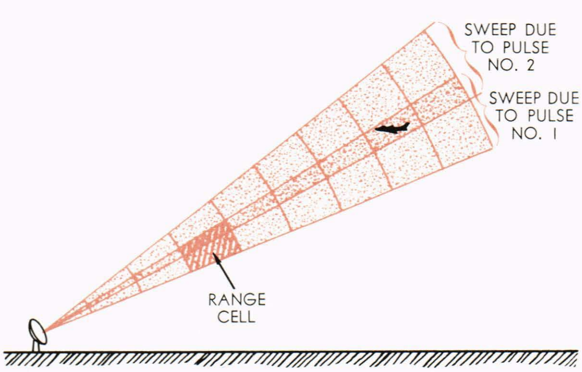


Fig. 1—Schematic representation of the division of space into range cells.

in the absence of targets, the probability is 50% that a false alarm will not occur. The corresponding probability of a false alarm occurring at each of the 10^8 opportunities is 6.93×10^{-9}.

BIAS LEVEL—The probability that noise alone will exceed a given bias level is obviously a function of the bias level. The nature of the function depends on the combined law of the detector and integrator and on the characteristics of the noise. The detector referred to here is the envelope detector, the output of which is a given function of the envelope of the carrier wave; this function is called the law of the detector. The incoherent integrator affects the statistical problem in the same way as the detector. The function of the signal voltage, which is integrated, is called the law of the integrator, e.g. the square of the pulse voltage might be integrated over N pulses. As long as the same weight is applied to each of the N pulses, the integrator is called linear, e.g. we could have a linear square-law integrator.

The solutions for the bias level obtained by Marcum are for the combined law of the detector and integrator. The solution obtainable rigorously is for a combined square law. This combination is usually thought of as a square-law detector coupled with a linear linear-law integrator.

The bias level for the combined square-law case, and for the assumed normal distribution of noise voltage, is given by

$$P_0^{\frac{1}{n'}} = \int_0^{Y_b} \frac{Y^{N-1} e^{-Y}}{(N-1)!} \, dY, \qquad (5)$$

and

$$Y = \sum_1^N y, \qquad (6)$$

where Y_b is the bias level normalized to root-mean-square noise, and y is the output of the detector normalized to root-mean-square noise.

The noise upon which Eq. (5) is based is representative of the actual noise that usually interferes with radar reception. Its mathematical representation states that the probability that a given voltage due to noise will occur is normally distributed; this is called Gaussian noise. It is also assumed that the power spectral density of the noise is constant; this is referred to as "white" noise. Mean-square noise is defined as

$$\psi_0 = \int_0^\infty w(f) \, df, \qquad (7)$$

where ψ_0 is mean-square noise (or average noise power), $w(f)$ is the power spectral density of the noise output of the receiver (assumed to be a linear filter), and f is the frequency. When white noise is passed through a linear filter, the power spectral density of the output is proportional to

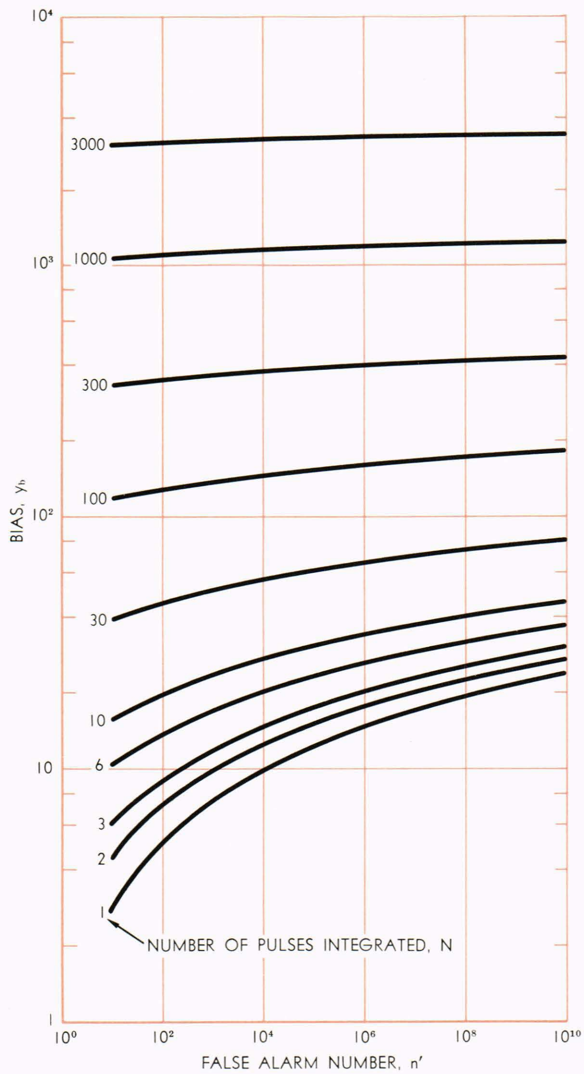


Fig. 2—The bias required on root-mean-square noise to establish a threshold.

the square of the filter's transfer function. Thus $w(f)$ is the square of the transfer function of the receiver multiplied by a constant to account for the noise power level. The relationships among Y_b, N, and n' are shown in Fig. 2 for $P_0 = 0.5$.

TARGET DETECTION—A functional block diagram of an automatic pulse radar may be drawn as in Fig. 3. The coherent integrator adds pulses in phase either at carrier frequency or at some intermediate frequency. The output is fed to the envelope detector and also to a device to obtain ψ_0 and take the square root; the value obtained for ψ_0 must not be unduly influenced by echoes. This value is multiplied by the factor Y_b to obtain the bias level. The integrated output of the detector is then compared with the bias level in the threshold device. If the integrated detector output is larger, the alarm indicating the probable presence of a target is sounded. Marcum's analysis applies only to that part of the functional block diagram downstream of the coherent integrator. If m pulses are integrated coherently, the analysis applies to $1/m$ of the transmitted pulses, each of which has m times the energy of a single transmitted pulse.

The probability of detecting a non-fluctuating target, i.e. the probability of sounding the alarm in the presence of signal and noise, is given by[2]

$$P_N(x, Y_b) = e^{-Nx} \sum_{k=0}^{\infty} \frac{(Nx)^k}{k!} \sum_{j=0}^{N-1+k} \frac{e^{-Y_b} Y_b^j}{j!}, \qquad (8)$$

where x is the signal-to-noise power ratio, and k and j are summation indices. Statisticians will immediately recognize the second summation of Eq. (8) as the Incomplete Gamma Function restricted to integers. Thus Eq. (8) is the sum of an infinite number of terms, each one of which is an Incomplete Gamma Function multiplied by a coefficient. Under some circumstances the conjugate form of Eq. (8) is also useful.

Since the relationships have been established among the bias level, the false-alarm number, and the number of pulses integrated by Eq. (5), the probability of detecting echoes in the presence of noise can now be related to the false-alarm number through Eq. (8). Representative samples of these relationships are shown in Fig. 4.

Swerling extended Marcum's square-law results to four different cases in which targets return echoes of fluctuating strength. Cases 1 and 2 apply to targets that can be represented as a number of independently fluctuating reflectors of about equal echoing area. Such is said to be the case for objects that are large compared to a wavelength and shaped not too much like a sphere. It is claimed that observed data on aircraft targets agree with the density distribution[3] assumed for Cases 1 and

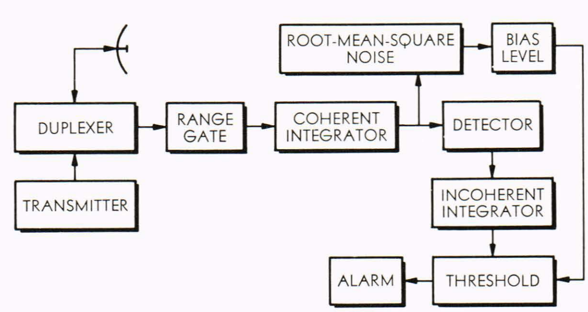


Fig. 3—Block diagram of an automatic pulse radar.

[2] The series representation was derived by R. G. Roll, APL, from the original characteristic functions by means of contour integration.

[3] This distribution is $w(x, \bar{x}) = (1/\bar{x})\exp(-x/\bar{x})$; $x \geqq 0$, where $\bar{x}$ is the average signal-to-noise ratio over all target fluctuations.

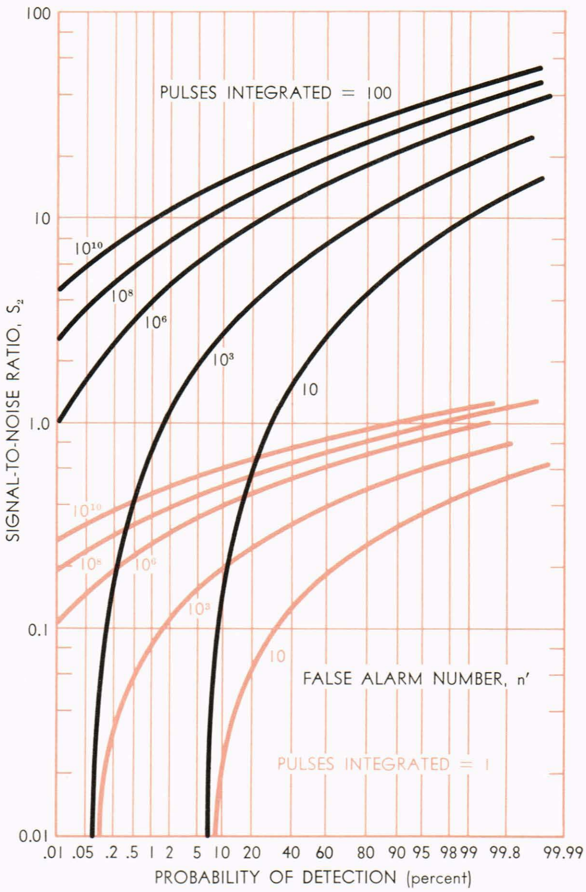


Fig. 4—Probability of detecting a non-fluctuating target with pulses integrated equal to 1 or 100.

2. Cases 3 and 4 apply to targets that can be represented as one large reflector, together with a number of small reflectors, or as one large reflector subject to small changes in orientation.[4]

Cases 1 and 3 apply when echo fluctuations occur from scan to scan. During a scan the pulse-to-pulse echo strength is assumed constant, i.e. fully correlated. Cases 2 and 4 apply when fluctuations occur from pulse to pulse, i.e. fully uncorrelated. The probability of detection for all four cases is expressed in terms of the Incomplete Gamma Function in a manner similar to Eq. (8). This greatly facilitates the computation of new data. Representative data computed for Swerling's four cases are shown in Figs. 5 and 6.

Range Equation—As mentioned above in connection with Eq. (1), the signal-to-noise power ratio is available by measuring signals and noise over the same period of time. Since the probability of detection is also a function of the signal-to-noise power ratio, all the parameters required for forming the range equation are available in consistent units. Noise power from such an unavoidable source as the receiver is usually normalized to a standard thermal noise power.

The range equation can now be written. It expresses, among other things, the relationship between the range and the probability that a detection will be made each time the range is sampled. These detections will include false alarms.

$$R^4 = \frac{m\,\widehat{P}_t\,G_t\,G_R\,\lambda^2\,\sigma\,L}{(4\pi)^3(KT\beta F)\;S(N, n', P_N)}, \qquad (9)$$

where $\widehat{P}_t$ is peak power in relation to the average transmitted power of the radar (actually the average power of each transmitted pulse),
σ is scattering cross section of the target,
K is Boltzmann's constant for thermal noise energy—1.38×10^{-23} joules/°K,
T is absolute temperature (I.R.E. standard is 290°K),
β is the receiver bandpass,

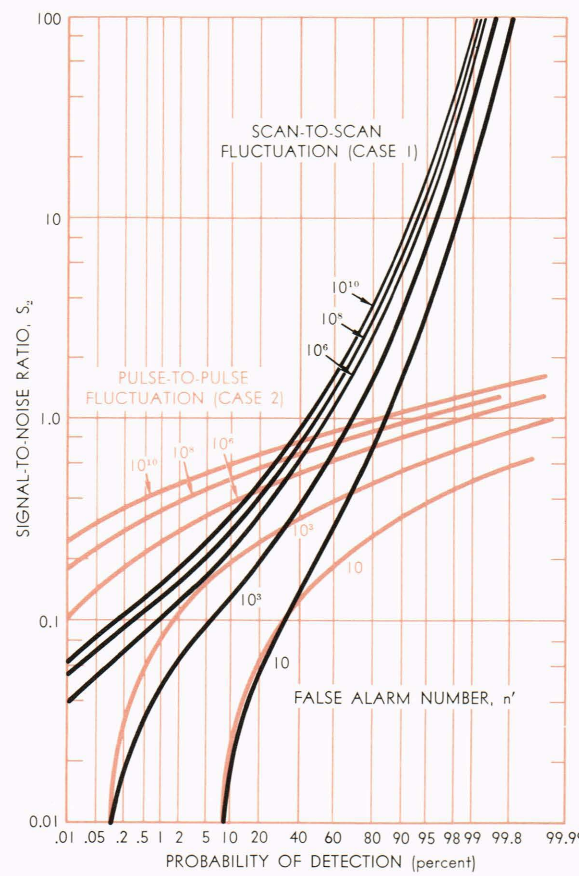


Fig. 5—Probability of detecting a fluctuating target, with pulses integrated equal to 100 and fluctuating reflectors of equal size.

[4] The assumed density distribution is $w(x, \bar{x}) = 4(x/\bar{x}^2)\exp(-2x/\bar{x})$; $x \geqq 0$.

F is the noise figure of the receiver, and
S is the signal-to-noise power ratio (given as x in Eq. (8) as a function of N, n', and P_N).

The numerator of Eq. (9), divided by $[(4\pi)^3 R^4]$, expresses the summed power of m reflected pulses measured at the input to the receiver. The term in brackets in the denominator expresses the noise power due to the usual source, i.e. the noise of the receiver referred to thermal noise. If there are present such other sources of noise as noise jammers, their power as measured at the input to the receiver must be added to the term in brackets. Equation (9) is written for a pulse radar. However, it can be made to apply to a CW radar by interpreting the samples of the continuous return as pulses and substituting average power for peak power.

Collapsing Loss—The function $S(N, n', P_N)$, which is shown in Figs. 4, 5, and 6, has been computed for conditions under which the radar set integrates the same number of noise pulses as signal pulses. If more noise than signal pulses are integrated, a greater signal-to-noise ratio is associated with any given probability of detection than is given by $S(N, n', P_N)$. A collapsing ratio is defined as

$$\rho = \frac{M + N}{N}, \qquad (10)$$

where ρ is the collapsing ratio and M is the additional number of noise pulses integrated.

The term "collapsing" stems from an association of this ratio with the superposition, or "collapsing," of data on a radar scope. A collapsing loss is also defined as

$$L_c = \frac{S_1}{S}, \qquad (11)$$

where L_c is the collapsing loss factor, S is the signal-to-noise ratio for $M = 0$, and S_1 is the signal-to-noise ratio for $M > 0$. In cases where collapsing loss applies, the denominator of Eq. (9) is multiplied by L_c. Note that in so doing $S(N, n', P_N)$ is merely changed to $S_1(M, N, n', P_{N+M})$.

Values of $S_1(M, N, n', P_{N+M})$ can be found from $S(N, n', P_N)$ through the use of ρ. First find the bias level from Fig. 2, using ρN for the parameter N noted on it. Then find a value for S for the desired probability of detection, using ρN for N in Figs. 4, 5, or 6. Multiply this value of S by ρ to obtain S_1. This value of S_1 can then be used directly in Eq. (9) or indirectly in Eq. (11). Symbolically when $P_{N+M} = P_N$,

$$S_1(M, N, n', P_{N+M}) = \rho S(\rho N, n', P_N). \qquad (12)$$

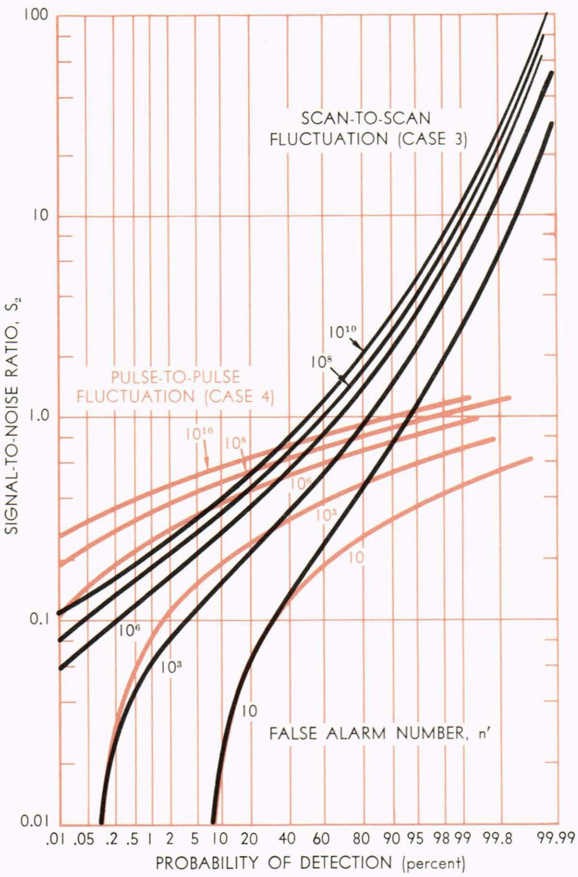


Fig. 6—Probability of detecting a fluctuating target, with pulses integrated equal to 100, large reflector plus several small reflectors, or large reflectors subject to small changes in orientation.

Conclusion

The detection probability is related to the bias through integrals whose values are very sensitive to small changes in the limits of integration, especially for a large number of pulses integrated incoherently. To obtain numerical values of detection probability as a function of false-alarm number, the number of significant figures required for the bias must be compatible with the sensitivity of the functions to be integrated. Accordingly, the bias Y_b was found from Eq. (5) using double precision arithmetic on the high-speed digital computer.[5] Various values of the false-alarm number, from 10 to 10^{10}, were used, and the standard value 0.5 was taken for the probability of not getting a false alarm in the false-alarm time.

The values of Y_b thus determined were used in Eq. (8) and in the equivalent equations for Swerling's cases. The number of pulses integrated,

[5] The computing program was written by G. T. Trotter, APL.

N, was varied from 1 to 3000. The signal-to-noise ratio was varied over a sufficient range to define a graph for a range of P_N between approximately 0.001 and 0.999.

The significant figures, which can be read from graphs of the data, are sufficient for most purposes. All of the data are graphed in Ref. 6, and if greater accuracy is required for any particular problem, the tabulated data are on file. These data were computed exactly to six significant decimal digits. The new data indicate that the accuracy of Marcum's data for non-fluctuating targets is at least as good as the accuracy of reading his graphs, which is poor for some ranges of the arguments. Swerling's data for all four cases, however, are quite approximate, especially when the number of pulses integrated is large. The exact data indicate lower probabilities of detection for the same signal-to-noise ratio.

[6] L. F. Fehlner, *Marcum's and Swerling's Data on Target Detection by a Pulsed Radar,* The Johns Hopkins University, Applied Physics Laboratory, TG-451, July 2, 1962.

An Electromechanical Time-Code Generator

A persistent problem in data recording, reduction, and analysis is that of linking significant events to their precise time of occurrence. To meet the problem, some data-handling specialists at APL time-code their records by means of commercial, electronic time-code generators—expensive devices well suited as fixed-installation equipment for use by highly skilled technicians. In actual use, a time-code generator records an analog reference time on tape simultaneously with a taped record of a particular sequence of events. The two taken as a unit constitute an accurate, time-referenced record, as, for example, of a missile firing.

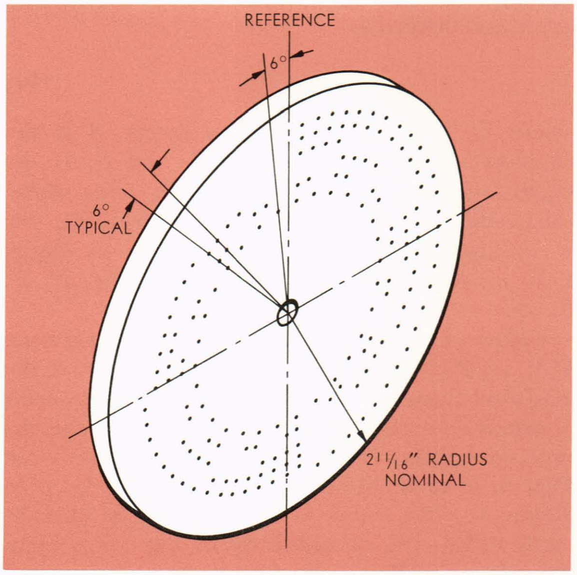
For mobile installations requiring simplicity of operation and minimum maintenance, however, the available commercial equipment is often too complex. To meet the demand for mobile equipment, W. E. Hanson* proposed that a photoelectric sensor could be used in an electromechanical device to provide a simple, inexpensive, chronometric generation of a code-pulse train. He then constructed a model that used a series of disks to interrupt light beams at regular intervals. By means of this well-known technique, timing pulse signals were produced by photodiodes that detected the interruptions of the light beam. A refinement of the model, suggested by E. H. Fischer,* was to use additional disks and to place them in pairs to strobe through a pattern of holes drilled in the disks. Stationary light beams would thereby be interrupted in a predetermined sequence to produce a timing code.

Just such a simple, rugged, and inexpensive device to provide a timing code for shipboard records of missile firings was needed at APL. Therefore, the time-code generator developed at APL was adapted to generate the Atlantic Missile Range (AMR), 13-bit, 1-pulse/sec code (it is equally adaptable to many other codes, including 100-pulse/sec codes).

The accuracy of the generator depends on a synchronous motor, which may be driven by 60-cycle AC power as in the AMR design. Increased accuracy may be obtained by using a tuning fork and power amplifier, or a piezoelectric crystal and power amplifier.

The coded photo readout is accomplished by using three disks, seconds, minutes, and hours respectively, in which holes are drilled at appropriate

*W. E. Hanson is a member of the Analog Playback Section, and E. H. Fischer is a member of the Electronic Development Project, Bumblebee Instrumentation Group.



The face of a representative disk, showing the pattern of holes drilled for minutes.